No M_3s and Maltsev imply majority

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Why?

- For fun.
- Ross Willard's project: A better algorithm for CSP with Maltsev.
- We want to understand the applicability of bounded width results outside of CSPs with bounded width.

Our world

- I CSP instance, all constraints unary or binary.
- All algebras are idempotent.
- Unary constraints . . . potatoes.
- Binary constraints . . . lines.
- Each potato is an algebra.
- Our polymorphisms are multisorted and have to preserve all constraints.

The power of consistency

- An instance of CSP is (1,1)-minimal (subdirect) if all constraints are subdirect.
- An instance of CSP is (2,3)-minimal if all constraints are subdirect and each line can be extended to a triangle.
- Ross: If I is a (2,3)-minimal instance with a multisorted Maltsev polymorphism and no M_3 in the congruence lattice of any potato, then I has a solution.
- Why?

Instancewide majority

- Ross's conjecture: Under conditions similar to the previous slide, I has an instance-wide majority polymorphism.
- This explains why consistency checking works.

Theorem

If I is a (1,1)-minimal instance with a multisorted Maltsev and no M_3 in the congruence lattice of any potato, then I has a multisorted polymorphism m such that for each $x, y \in B_i$ we have

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x.$$

The relation with relations

- How to construct m?
- We generalize eg. Valeriote, Horowitz.
- We have m if and only if for every i, j, k and every $a, a' \in A_i$, $b, b' \in A_j$, $c, c' \in A_k$ we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathrm{Sg}_{\mathbf{A}} \left(\begin{pmatrix} a' \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b' \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c' \end{pmatrix} \right)$$

Relations definable in I

- The relational clone of I is also multisorted.
- We can treat pp-defined relations as CSP instances with each variable either free or (existentially) quantified.
- Main insight: Under the assumptions of the theorem, any ternary relation on I can be drawn as an almost triangle between the three free variables.
- This rules out the bad relation R and we win.

Where to go from here?

- The congruence lattice approach to CSP instances gives results (ask Libor), so it makes sense to push in this direction.
- Replace Maltsev by cube term?
- Better algorithm for all I with Maltsev? (Ask Ross.)
- Improving Miklos' "Tree on top of Maltsev"?
- Is there a common generalization of this and "Maltsev implies majority" for digraphs?

Thank you for your attention.