Solving edge CSP with even delta-matroid constraints

Alexandr Kazda, Vladimir Kolmogorov, and Michal Rolínek

June 15th, 2017
AAA94 + NSAC 2017

Boolean Constraint Satisfaction Problem

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- ... to which we want to assign values 0 or 1 ...
- ... so that a set $\mathcal{C}$ of constraints is satisfied.
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- Goal: Find $f\colon V \to \{0,1\}$ that is a perfect matching:

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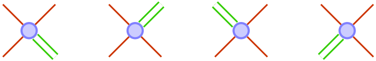
$\forall C \in \mathcal{C}$ we have 

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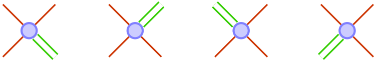
$\forall C \in \mathcal{C}$ we have 

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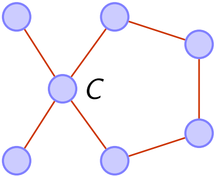
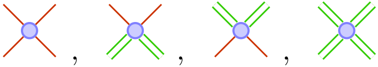


$C = \Big\{$  $\Big\}$

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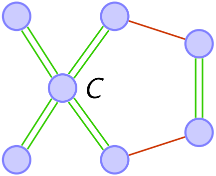
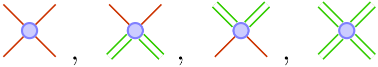


$C = \{$  $\}$

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- A (nonempty) relation $M \subset \{0,1\}^n$ is a Δ-matroid if it satisfies a certain exchange axiom.
- Previous algorithms for special classes of Δ-matroids: co-independent (Feder, 2001), compact (Istrate, 1997), local (Dalmau and Ford, 2003), binary (Geelen, Iwata and Murota, 2003; Dalmau and Ford, 2003).
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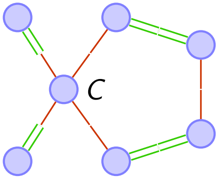
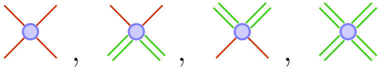


$C = \{$  $\}$

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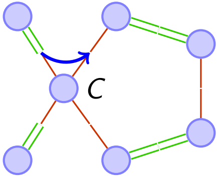
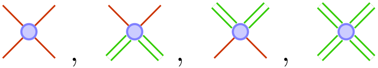


$C = \{$  $\}$

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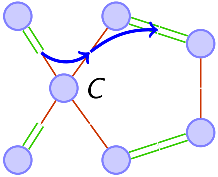
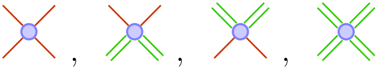


$C = \{$  $\}$

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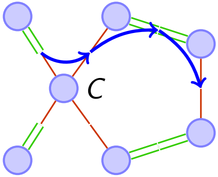
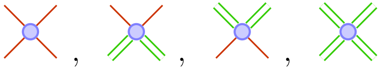


$C = \{$  $\}$

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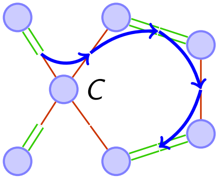
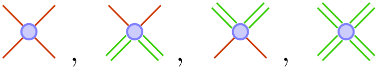


$C = \{$  $\}$

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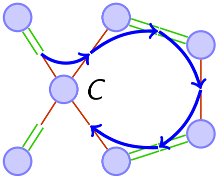
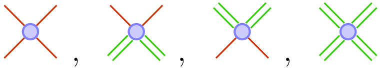


$C = \{$  $\}$

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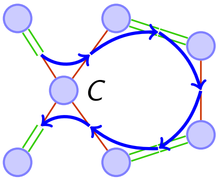

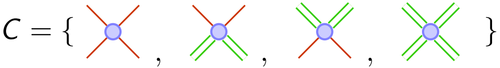

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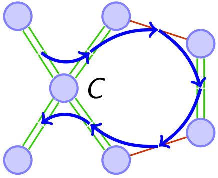
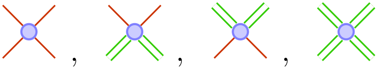


$C = \{$  $\}$

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- If we can reach a variable from both sides, we have a blossom.
- If that happens we contract the blossom and recursively solve a "smaller" edge CSP instance.
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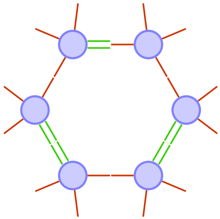

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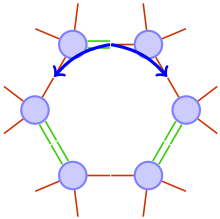

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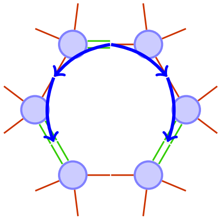

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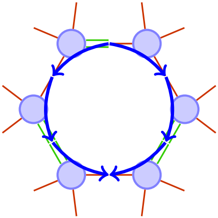

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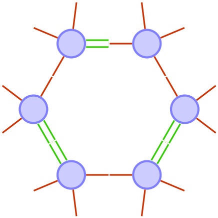

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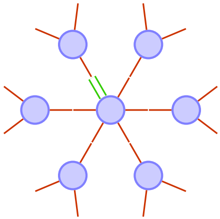

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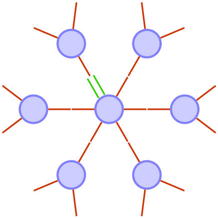

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- Our algorithm $\Rightarrow$ Dichotomy for planar Boolean CSP.

Open problems

- How to find a solution of minimal cost?
- We can handle effectively coverable Δ-matroids $\supseteq$ previously known tractable classes.
- Algorithm for general Δ-matroids?
- Generalization to value sets larger than 2?
- Where is the algebraic approach hiding?!

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Thank you for your attention.