How to decide absorption

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What is absorption?

Definition (Libor Barto, Marcin Kozik)

Let $\mathbf{B} \leq \mathbf{A}$ be algebras. We say that $\mathbf{B}$ absorbs $\mathbf{A}$ if there exists a term t in $\mathbf{A}$ such that for any $b_1, \ldots, b_n \in B, a \in A$ we have:

$$\begin{aligned} t(a, a, a, \ldots, a) &= a \\ t(a, b_2, b_3, \ldots, b_{n-1}, b_n) &\in B \\ t(b_1, a, b_3, \ldots, b_{n-1}, b_n) &\in B \\ &\vdots \\ t(b_1, b_2, b_3, \ldots, b_{n-1}, a) &\in B \end{aligned}$$

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Ok, but what *is* absorption?

- If 0 is the minimal element of a finite semilattice $(L, \wedge)$, then $\{0\}$ absorbs L; absorption term is $t(x_1, x_2) = x_1 \wedge x_2$.
- If $\mathbf{A}$ is an algebra with a majority term m, then every singleton is an absorbing subalgebra; absorption term is m.
- If $\mathbf{A}$ is any algebra, then always $\mathbf{A} \trianglelefteq \mathbf{A}$.
- If $\mathbf{A}$ is an abelian group, then $\mathbf{A}$ has no proper absorbing subalgebra.

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Deciding absorption

- Let $\mathbf{A}$ be an idempotent finite algebra. Then $\mathbf{A}$ has an NU term iff every singleton $\{a\}$ absorbs $\mathbf{A}$.
- Miklós Maróti, Libor Barto, Dmitriy Zhuk: We can decide whether a finite algebra $\mathbf{A}$ has an NU term.
- Problem: Given $\mathbf{B} \leq \mathbf{A}$, can we decide if $\mathbf{B} \trianglelefteq \mathbf{A}$?
- Libor Barto, Jakub Bulín: Yes, if $\mathbf{A}$ is finitely related.
- What about if $\mathbf{A}$ is given by a finitely many operations instead?

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Blockers

- Let $\mathbf{B} \trianglelefteq \mathbf{A}$ with absorption term t.
- We call (C, D) a blocker for $\mathbf{B}$ if
  - $\emptyset \neq D \subset C$,
  - $C \cap B \neq \emptyset$,
  - $D \cap B = \emptyset$,
  - $\{(x_1, \ldots, x_n) \in C^n : \exists i, x_i \in D\} \leq A^n$ for every $n \in \mathbb{N}$.
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No blockers $\Rightarrow$ absorption?

- Given idempotent $\mathbf{A}$ with finitely many operations, we can test if there are no blockers for $\mathbf{B}$.
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Jónsson absorption

- Weaker notion of absorption inspired by terms for congruence distributivity.
- Let $\mathbf{B} \leq \mathbf{A}$. Then $\mathbf{B} \trianglelefteq_J \mathbf{A}$ if there exist idempotent terms $d_0, d_1, \ldots, d_n$ such that:

$$\begin{aligned} \forall i = 0, \ldots, n,\ d_i(B, A, B) &\subset B \\ d_0(x, y, z) &= x \\ d_i(x, y, y) &= d_{i+1}(x, y, y) \text{ for } i \text{ even} \\ d_i(x, x, y) &= d_{i+1}(x, x, y) \text{ for } i \text{ odd} \\ d_n(x, y, z) &= z. \end{aligned}$$

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Putting it all together

Theorem

Let $\mathbf{A}$ *be a finite idempotent algebra,* $\mathbf{B} \leq \mathbf{A}$*. Then* $\mathbf{B} \trianglelefteq \mathbf{A}$ *iff there is no blocker for* $\mathbf{B}$ *and* $\mathbf{B} \trianglelefteq_J \mathbf{A}$*.*

Corollary

We can decide $\mathbf{B} \trianglelefteq \mathbf{A}$ *algorithmically for idempotent algebras.*

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Nonidempotent algebras

- If **A** is not idempotent, we would also like to decide to absorption.
- Problem with taking the idempotent reduct: We might lose the generators of the clone of **A**.
- Imitating some of Dmitriy Zhuk's ideas should give us an algorithm anyway. . .

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Thank you for your attention.