Random CSP complexity

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Outline

Our problems

- For a fixed digraph G, we would like to decide whether there exists a homomorphism from some input digraph H to G.
- This is the *Digraph Homomorphism Problem*. Its complexity depends on the choice of G.
- The *Homomorphism Extension Problem* is similar, only we want to know if we can extend some partial homomorphism $f : H \to G$ to a homomorphism.

CSP

- Both problems can be described in the language of the *Constraint Satisfaction Problem*.
- We look for a map $h : V(H) \to V(G)$ that satisfies constraints.
- Edge constraint on the pair $u, v \in V(H)$ is "$(h(u), h(v)) \in E(G)$".
- For any $w \in V(G)$ we have the *constant constraint* which for $v \in V(H)$ says "$h(v) = w$".
- We can also combine constraints.

Random digraph

- A random digraph on n vertices will be a digraph where each edge exists with a probability $p \in (0,1)$. We fix $p = 1/2$ for simplicity.
- How hard is the Homomorphism Extension Problem for G large random digraph?

Intermezzo

Theorem (Hell, Nešetřil, '90)

Let G be an undirected graph. Then the Homomorphism Problem of G is polynomially solvable if G is bipartite or contains a loop and NP-complete otherwise.

Random Digraph Homomorphism Complexity

- If G is a large random digraph without loops allowed then Digraph Homomorphism Problem for G is almost surely NP-complete.
- If G is a large random digraph with loops allowed then Digraph Homomorphism Problem for G is almost surely polynomial.
- Another approach: random G has only projections as its polymorphisms (see Nešetřil and Luczak).

Homomorphism Extension Problem is Hard

- We prove that extending homomorphisms to G is hard even if there are loops in G.
- For $v_1, \dots, v_k$ vertices of G, denote $S_{v_1,\dots,v_k} = \{w \in V(G) : (v_1, w), \dots, (v_k, w) \in E(G)\}$.
- It is easy to construct a constraint "$h(u) \in S_{v_1,\dots,v_k}$".
- The set $S_{v_1,\dots,v_k}$ is a *subalgebra* of $V(G)$.

Subalgebras are our friends

- If we find in G many disjoint three element subalgebras, we have won.
- At least one of these subalgebras will almost surely contain K_3 and so we can reduce 3-coloring to our homomorphism extension problem.
- But can we do that?

Taming the random digraph

- Let us choose $k = \log(n/3)$ and order the vertices of G.
- We now keep constructing $S_{v_1,\dots,v_k}$ until we produce a subalgebra.
- We have a lot of shots, since $\log n \ll n$.
- We are careful not to touch the insides of subalgebras.

Finishing the proof

- We have shown that G has almost surely k disjoint three element subalgebras for *any* k.
- Probability that there is no K_3 subalgebra is then at most $\left(1 - 1/2^9\right)^k$.
- Let K be the event "G contains a K_3 induced by some subalgebra".
- We have

$$\liminf_{n\to\infty} \mathrm{P}(K) = 1$$

- Therefore, our problem is almost surely NP-complete.

Thanks for your attention.