Universal quadratic forms over number fields

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Outline

Universal quadratic forms

Quadratic form $Q(X_1, \ldots, X_r) = a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 + \ldots$
with $a_{ij} \in \mathbb{Z}$

Which integers are represented?

A quadratic form is *universal* if it represents all positive integers.

Many indefinite forms, eg. $X^2 - Y^2 - dZ^2$ with $4 \nmid d$.

More interesting are positive definite forms.

- No universal positive form in 3 variables
- Lagrange (1770): $X^2 + Y^2 + Z^2 + T^2$
- Ramanujan, Dickson (1916): classified quaternary universal positive diagonal forms, eg. $X^2 + 2Y^2 + 4Z^2 + dT^2$ with $d \leq 14$

290 Theorem (Bhargava–Hanke 2011)

A positive definite quadratic form over $\mathbb{Z}$ is universal $\iff$ it represents $1, 2, 3, \ldots, 290$.

Quadratic forms over number fields

K = totally real number field
$\mathcal{O}$ = ring of integers in K

Quadratic form $Q(X_1, \dots, X_r) = a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 + \dots$ with $a_{ij} \in \mathcal{O}$ is *universal* if

- it is totally positive definite and
- represents all totally positive elements of $\mathcal{O}$

How about sum of squares $X_1^2 + X_2^2 + \cdots + X_r^2$?
Siegel (1945): Universal only for

- $K = \mathbb{Q}$ $\quad r = 4$
- $K = \mathbb{Q}(\sqrt{5})$ $\quad r = 3$

More general universal forms exist over any K

Universal forms

Questions.

1. Kitaoka's Conjecture: There are only finitely many K with universal *ternary* form
2. How does the minimal number of variables r depend on K?
3. Is there a variant of 290 Theorem?

Previous results.

- Earnest–Khosravani (1997): no ternary universal forms over fields of odd degree
- Chan–Kim–Raghavan (1996): Determined all ternary universal forms over quadratic fields $\mathbb{Q}(\sqrt{D})$ (only $D = 2, 3, 5$)
- Kim (1999): 8-ary universal form over each $\mathbb{Q}(\sqrt{n^2 - 1})$
- Kim-Kim-Park (2021): only finitely many $\mathbb{Q}(\sqrt{D})$ admit 7-ary universal forms

Outline

Real quadratic fields

$D > 1$ squarefree, $D \equiv 2, 3 \pmod 4$
$K = \mathbb{Q}(\sqrt{D})$
$\mathcal{O} = \mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$

Want to represent *totally positive* elements:

$$\mathcal{O}^+ = \{\alpha \in \mathcal{O} \mid \alpha = a + b\sqrt{D} > 0, \alpha' = a - b\sqrt{D} > 0\}.$$

Notation: $\alpha \succ \beta$ iff $\alpha - \beta \in \mathcal{O}^+$

Theorem (Blomer-K 2015, K 2016)

For each M there are infinitely many $\mathbb{Q}(\sqrt{D})$ with no universal M-ary form.

Want to explain main ideas behind proof

Tool 1: Indecomposable elements

$\alpha \in \mathcal{O}^+$ is *indecomposable* if $\alpha \neq \beta + \gamma$ for $\beta, \gamma \in \mathcal{O}^+$

Seems to be the key notion for studying universal forms!

Every unit $\varepsilon \in \mathcal{O}^+$ is indecomposable:

$$1 = N(\varepsilon) = N(\beta+\gamma) = (\beta+\gamma)\cdot(\beta+\gamma)' \succ$$
$$\succ \beta\beta' + \gamma\gamma' = N(\beta) + N(\gamma) \geq 2$$

Why useful?
$Q(X_1, \dots, X_r) = a_1X_1^2 + a_2X_2^2 + \cdots + a_rX_r^2$ universal diagonal form
$\alpha = a_1x_1^2 + a_2x_2^2 + \cdots + a_rx_r^2$ indecomposable
Thus $\alpha = a_ix_i^2$ is essentially one of the coefficients!

Tool 2: Continued fractions

Periodic continued fraction

$$\sqrt{D} = [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, 2u_0}] = u_0 + \cfrac{1}{u_1 + \frac{1}{u_2 + \cdots}}.$$

Convergents $\frac{p_i}{q_i} := [u_0, u_1, u_2, \ldots, u_i]$ to the continued fraction give

- good approximations $\frac{p_i}{q_i}$ to $\sqrt{D}$ and
- indecomposables, eg $\alpha_i := p_i + q_i\sqrt{D}$ (for odd i)

Recall: α_{s-1} = fundamental unit

Explicitly,
$\alpha_{i,r} := \alpha_i + r\alpha_{i+1}$ for odd i, $0 \le r < u_{i+2}$
are all indecomposables

Theorem (Hejda-K 2020)

The additive semigroup $\mathcal{O}^+(+)$ uniquely determines the real quadratic field $K = \mathbb{Q}(\sqrt{D})$.

Tool 3: Minimal vectors of $\mathcal{O}$-lattices

Quadratic $\mathcal{O}$-lattice (L, Q):

- $L \simeq \mathcal{O}^r$
- $Q : L \to \mathcal{O}$ totally positive quadratic form on L

Corresponds to usual quadratic forms, but don't need to worry about change of variables

In $\mathbb{Z}$-lattice, *minimal vectors* are important, ie
$v \in L$ with minimal $Q(v) > 0$.
Eg. *kissing number* = maximal possible number of minimal vectors in L of given rank r

How to define minimal vectors in general?

Tool 3: Minimal vectors of $\mathcal{O}$-lattices 2

How to define "minimal vectors" in general?
Compose with trace!
$\mathrm{Tr} \circ Q : L \to \mathbb{Z}$

More generally,
codifferent $\mathcal{O}^\vee := \{\delta \in K : \mathrm{Tr}(\delta\mathcal{O}) \subset \mathbb{Z}\}$
For $\delta \in \mathcal{O}^{\vee,+}$ and $0 \neq v \in L$ have $\mathrm{Tr}(\delta Q(v)) \in \mathbb{Z}_{>0}$

$\Rightarrow$ can take vectors v with $\mathrm{Tr}(\delta Q(v))$ minimal,
ideally $\mathrm{Tr}(\delta Q(v)) = 1$

v "minimal" (for some δ) $\Rightarrow$ $Q(v)$ indecomposable
Converse holds in quadratic fields (K-Tinková), not in general

Summary

Theorem (Blomer-K 2015, K 2016)

For each M there are infinitely many $\mathbb{Q}(\sqrt{D})$ with no universal M-ary form.

Proof by K-Tinková (2021):

Take $\sqrt{D} = [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, 2u_0}]$

For fixed odd i, have $u := u_{i+2}$ indecomposables $\alpha_{i,r}$
and uniform δ such that $\mathrm{Tr}(\delta\alpha_{i,r}) = 1$ for all r

Let (L, Q) be universal $\mathcal{O}$-lattice of rank R
$\implies$ have $\alpha_{i,r} = Q(v_r)$ for $v_r \in L$

$\mathrm{Tr}(\delta Q)$ gives $\mathbb{Z}$-lattice of rank $2R$
with u vectors of length 1 (corresponding to v_r)

Kissing number estimates $\implies R \geq \sqrt{u}/2$

Suffices to find D with eg u_1 large to finish proof

Outline

Constructing universal forms

$$\sqrt{D} = [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, 2u_0}]$$

S = (finite) set of all indecomposables σ
up to multiplication by squares of units

$$U_D := \#S = \begin{cases} 2(u_1 + u_3 + \cdots + u_{s-1}) & \text{if } s \text{ is even} \\ 2u_0 + u_1 + u_2 + \cdots + u_{s-1} & \text{if } s \text{ is odd} \end{cases}$$

Theorem (Kim 1999, Blomer-K 2018, K-Tinková 2021)

$$\sum_{\sigma \in S} \sigma \left(x_{1j}^2 + x_{2j}^2 + x_{3j}^2 + x_{4j}^2 + x_{5j}^2 \right)$$

is universal and has $5U_D$ *variables*

Sums of coefficients

$$U_D = \#S = \begin{cases} 2(u_1 + u_3 + \cdots + u_{s-1}) & \text{if } s \text{ is even} \\ 2u_0 + u_1 + u_2 + \cdots + u_{s-1} & \text{if } s \text{ is odd} \end{cases}$$

How large is U_D? Can we get a corresponding lower bound on rank?

Theorem (Blomer-K 2018)

- *Each diagonal universal quadratic form has at least CU_D^* variables, where U_D^* is the same sum of u_i as U_D, but only over coefficients $u_i \geq D^{1/8+\varepsilon}$.*
- $U_D \leq c\sqrt{D}(\log D)^2$
- *If s is odd, then $U_D^* \geq \sqrt{D}$.*

Expect (very imprecisely) $U_D \approx U_D^* \approx \frac{\sqrt{D}\log D}{h}(+\sqrt{D}$ if s is odd)

Indecomposables roughly correspond to principal ideals of norm $\ll \sqrt{D}$, should have $\sqrt{D}/h$ of them

Higher degrees

K totally real number field of degree n over $\mathbb{Q}$
$m(K)$ = minimal number of variables of universal form

Complicated, because continued fractions don't work.
Partial results on large number of variables:

- Yatsyna (2019): $n = 3$
- K-Svoboda (2019): $n = 2^k$
- K (2021): n divisible by 2 or 3

Theorem (K-Tinková 2020)

Let $a \geq -1$ and consider simplest cubic field $K = \mathbb{Q}(\alpha)$ with $\alpha^3 - a\alpha^2 - (a+3)\alpha - 1 = 0$ (and $a^2 + 3a + 9$ squarefree). Then

$$\frac{\sqrt{a^2+3a+8}}{3\sqrt{2}} < m(K) \leq 3(a^2+3a+6).$$

Outline

Lifting problem

$\mathbb{Z}$-form = positive definite form with $\mathbb{Z}$-coefficients

$$Q(X_1, \ldots, X_r) = a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 + \ldots \text{ with } a_{ij} \in \mathbb{Z}$$

Question. (The lifting problem)
Can a $\mathbb{Z}$-form be universal over K?

Siegel: sum of squares NOT, unless $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$

Theorem (K-Yatsyna 2021)

K = totally real number field of degree n with a universal $\mathbb{Z}$-form.

- *If $n = 2$, then $K = \mathbb{Q}(\sqrt{5})$.*
- *If $n = 3, 4, 5, 7$ and K has principal codifferent ideal, then $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ (where $\zeta_7 = e^{2\pi i/7}$)*
- *$X^2 + Y^2 + Z^2 + W^2 + XY + XZ + XW$ is universal over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$*

Tool 4: Zeta functions

Dedekind zeta-function:

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s} = \prod_P \frac{1}{1 - N(P)^{-s}} \text{ for real part of } s > 1$$

Siegel (1969): formula for $\zeta_K(2)$ as a sum of trace 1 elements (for $n = 2, 3, 4, 5, 7$):

$$\zeta_K(2) = (-1)^n \pi^{2n} 2^{n+2} b_n |\Delta_K|^{-3/2} \sum_{\alpha \in \mathcal{O}_K^{\vee,+}, \mathrm{Tr}(\alpha)=1} \sigma((\alpha)(\mathcal{O}_K^{\vee})^{-1})$$

Used to prove our Theorem (K-Yatsyna)
$X^2 + Y^2 + Z^2 + W^2 + XY + XZ + XW$ over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ is the only form in its genus

In higher degrees n, $\zeta_K(2)$ involves also elements of trace $2, 3, \ldots$

Weak Lifting Problem

Theorem (K-Yatsyna 2021)

For each m, n, there are at most finitely many totally real number fields K of degree n with a $\mathbb{Z}$-form that represents all of $m\mathcal{O}^+$.

Sketch of proof.

$||\alpha|| = \max_\sigma(|\sigma(\alpha)|)$ for $\sigma : K \hookrightarrow \mathbb{R}$

- There is k such that kQ is sum of squares of linear forms
  $\implies$ suffices to prove for sum of squares $X_1^2 + \cdots + X_r^2$
- $E \subset K$ subfield generated by all elements $\alpha \in K$ s.t. $||\alpha|| < X$
  only finitely many such $\alpha \implies$ finitely many possible E
- can assume $E \subsetneq K$;
  let $\beta \in \mathcal{O} \setminus E$ have smallest $||\beta|| = Y \geq X$
- $mk(\beta + N) = \alpha_1^2 + \cdots + \alpha_r^2$, some $\alpha_i \notin E$
- $3mkY > ||mk(\beta + N)|| \geq ||\alpha_i||^2 \geq Y^2$, contradiction

Kitaoka's Conjecture

Conjecture. There are only finitely many K with universal *ternary* form

Theorem (K-Yatsyna 2021, "Weak Kitaoka")

For each n, there are at most finitely many totally real number fields K of degree n with universal ternary *form Q.*

Basic idea.

Q represents $1, 2, \ldots, 290$

Excluding finitely many K,

this representation is over a proper subfield $E_1 \subsetneq K$.

Consider corresponding subform Q_1 over E_1

Two possibilities:

a) Q_1 is universal ternary over $E_1 \implies$ can use Weak Lifting Theorem

b) else consider representation of "290-set" for E_1

again over a proper subfield $E_1 \subsetneq E_2 \subsetneq K$

Continue to get finite list of possibilities for K

Thanks

Thanks for your attention!

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Joyeux anniversaire,
professeur Waldschmidt !