Lifting problem for universal quadratic forms

Vítězslav (Vita) Kala
(joint work with Pavlo Yatsyna)

Charles University, Prague

sites.google.com/site/vitakala/

Supported by Czech Science Foundation GAČR, grant 17-04703Y

Journées Arithmétiques
July 4, 2019

Outline

Universal forms over $\mathbb{Z}$

A quadratic form with integral coefficients is *universal* if it represents all positive integers.
Many indefinite forms, eg. $X^2 - Y^2 - dZ^2$ with $4 \nmid d$.
Our topic are positive definite forms.

- No ternary universal positive forms
- Lagrange (1770): $X^2 + Y^2 + Z^2 + T^2$
- Ramanujan, Dickson (1916): classified quaternary universal positive forms, eg. $X^2 + 2Y^2 + 4Z^2 + dT^2$ with $d \leq 14$

Theorem (Conway-Schneeberger, Bhargava-Hanke)

A diagonal positive form over $\mathbb{Z}$ is universal iff it represents $1, 2, 3, \ldots, 15$.
An integral positive form over $\mathbb{Z}$ is universal iff it represents $1, 2, 3, \ldots, 290$.

Outline

Overview

Totally positive universal forms over (totally real) number fields.
Not much known, eg.

- Siegel (1945): If K totally real and sum of r squares is universal, then $K = \mathbb{Q}$ $(r = 4)$ or $K = \mathbb{Q}(\sqrt{5})$ $(r = 3)$.
- Chan-Kim-Raghavan (1996): Determined all ternary universal forms over $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{5})$
- Kim (1999): 8-ary universal form over $\mathbb{Q}(\sqrt{n^2-1})$, but only finitely many $\mathbb{Q}(\sqrt{D})$ admit *diagonal* 7-ary universal forms.
- Blomer-K (2015), K (2016): For each M, there are infinitely many real fields $\mathbb{Q}(\sqrt{D})$ with no universal M-ary forms.
- Yatsyna (2019), K-Svoboda (2019): Extended to certain higher degree fields.
- Blomer-K (2018): Estimated number of variables of *diagonal* universal forms over $\mathbb{Q}(\sqrt{D})$ in terms of continued fraction for $\sqrt{D}$.

Definitions

Notation:

- K totally real number field *(always)*
- $\mathcal{O}_K$ = ring of integers
- $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_d : K \hookrightarrow \mathbb{R}$ real embeddings
- norm $N(\alpha) = \sigma_1(\alpha) \cdots \sigma_d(\alpha)$ and trace $\mathrm{Tr}(\alpha) = \sigma_1(\alpha) + \cdots + \sigma_d(\alpha)$
- $\alpha \succ \beta$ iff $\sigma_i(\alpha) > \sigma_i(\beta)$ for all $1 \leq i \leq d$
- $\mathcal{O}_K^+ = \{\alpha \in \mathcal{O}_K \text{ with } \alpha \succ 0\}$

A quadratic form

$$Q(X_1, \ldots, X_r) = \sum_{1 \leq i \leq j \leq r} a_{ij} X_i X_j \text{ with } a_{ij} \in \mathcal{O}_K \text{ is}$$

- *totally positive* if all values $Q(x_1, \ldots, x_r) \succ 0$, $Q(0, \ldots, 0) = 0$
- *universal* if for each $\alpha \in \mathcal{O}_K^+$ there are $x_i \in \mathcal{O}_K$ with $Q(x_1, \ldots, x_r) = \alpha$

$\mathbb{Z}$-forms

$\mathbb{Z}$-form $=$ positive form with $\mathbb{Z}$-coefficients

$$Q(X_1, \ldots, X_r) = \sum_{1 \le i \le j \le r} a_{ij} X_i X_j \text{ with } a_{ij} \in \mathbb{Z}$$

Can a $\mathbb{Z}$-form be universal over K?

Siegel: sum of squares NOT, unless $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$

Theorem (K-Yatsyna 2019)

K = totally real number field of degree d with a universal $\mathbb{Z}$-form.

- *If $d = 2$, then $K = \mathbb{Q}(\sqrt{5})$.*
- *If $d = 3, 4, 5, 7$ and K has principal codifferent ideal $\mathcal{O}_K^\vee$, then $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.*
- *$X^2 + Y^2 + Z^2 + W^2 + XY + XZ + XW$ is universal over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$*

$\mathcal{O}_K^\vee := \{\delta \in K : \mathrm{Tr}(\delta \mathcal{O}_K) \subset \mathbb{Z}\}$

Outline

Indecomposable elements

$\alpha \in \mathcal{O}_K^+$ is *indecomposable* iff $\alpha \neq \beta + \gamma$ for $\beta, \gamma \in \mathcal{O}_K^+$
Seems to be the key tool for studying universal forms!

Every unit $\varepsilon \in \mathcal{O}_K^+$ is indecomposable:

$$1 = N(\varepsilon) = N(\beta + \gamma) = \sigma_1(\beta + \gamma) \cdots \sigma_d(\beta + \gamma) \succ$$
$$\succ \sigma_1(\beta) \cdots \sigma_d(\beta) + \sigma_1(\gamma) \cdots \sigma_d(\gamma) = N(\beta) + N(\gamma) \geq 2$$

Similarly, if $N(\alpha) < 2^d$, then α is indecomposable.

Why useful?
$Q(X_1, \dots, X_r) = a_1 X_1^2 + \cdots + a_r X_r^2$ universal diagonal form
$\alpha = a_1 x_1^2 + \cdots + a_r x_r^2$ indecomposable
Thus $\alpha = a_i x_i^2$ is essentially one of the coefficients!

Minimal vectors of $\mathcal{O}_K$-lattices

Quadratic $\mathcal{O}_K$-lattice (L, Q):

- $L \simeq \mathcal{O}_K^r$
- $Q : L \to \mathcal{O}_K$ totally positive quadratic form on L

Corresponds to usual quadratic forms, but don't need to worry about changes of variables

In $\mathbb{Z}$-lattice, *minimal vectors* are very important, ie.
$v \in L$ with minimal $Q(v) > 0$.
Eg. *kissing number* = maximal possible number of minimal vectors in L of given rank r

How to define minimal vectors in general?

Minimal vectors of $\mathcal{O}_K$-lattices 2

How to define "minimal vectors" in general?
Compose with trace!
$\mathrm{Tr} \circ Q : L \to \mathbb{Z}$

More generally,
codifferent $\mathcal{O}_K^\vee := \{\delta \in K : \mathrm{Tr}(\delta \mathcal{O}_K) \subset \mathbb{Z}\}$
For $\delta \in \mathcal{O}_K^\vee$ and $0 \neq v \in L$ have $\mathrm{Tr}(\delta Q(v)) \in \mathbb{Z}_{>0}$

$\Rightarrow$ can take vectors v with $\mathrm{Tr}(\delta Q(v))$ minimal,
ideally $\mathrm{Tr}(\delta Q(v)) = 1$

v "minimal" (for some δ) $\Rightarrow$ $Q(v)$ indecomposable
Converse holds in quadratic fields (Tinková), not in general

Tensor product

Q is $\mathbb{Z}$-form $\Rightarrow$ $\mathrm{Tr}(\delta Q(v))$ is tensor product
Definition:

- $(L_1, Q_1), (L_2, Q_2)$ two quadratic $\mathbb{Z}$-lattices
- $(L_1 \otimes L_2, Q_1 \otimes Q_2)$ is defined as quadratic $\mathbb{Z}$-lattice by:
- $L_1 \otimes L_2$ has basis $e_i \otimes f_j$ where e_i, f_j bases of L_1, L_2
- $(Q_1 \otimes Q_2)(v_1 \otimes v_2) = Q_1(v_1)Q_2(v_2)$

$T_\delta(x) := \mathrm{Tr}(\delta x^2)$ quadratic form on $\mathcal{O}_K$
Fixing integral basis for $\mathcal{O}_K$, have $\mathcal{O}_K \simeq \mathbb{Z}^d$ and
$(\mathcal{O}_K, T_\delta)$ is a $\mathbb{Z}$-lattice of rank d

Tensor product 2

Proposition

Q is $\mathbb{Z}$-form of rank r,
$\delta \in \mathcal{O}_K^\vee$.
Then $(\mathcal{O}_K^r, \mathrm{Tr}(\delta Q)) \simeq (\mathcal{O}_K \otimes \mathbb{Z}^r, T_\delta \otimes Q)$.

$\Rightarrow$ want to understand minimal vectors of tensor products!

Definition

L = $\mathbb{Z}$-lattice of rank r
L is of E-type, if for each $\mathbb{Z}$-lattice M
all the minimal vectors of $L \otimes M$ are split,
i.e., $v \otimes w$ with $v \in L, w \in M$.

Kitaoka: L is of E-type if $r \leq 43$ or $\min L \leq 3$
T_δ has rank d, and so OK when $d \leq 43$

Outline

Quadratic fields

Theorem (K-Yatsyna 2019)

K = totally real number field of degree 2 that admits a universal $\mathbb{Z}$-form. Then $K = \mathbb{Q}(\sqrt{5})$.

Maass: $K = \mathbb{Q}(\sqrt{5}) \Rightarrow X^2 + Y^2 + Z^2$ is universal

Idea of proof

Assume Q is universal $\mathbb{Z}$-form over $K = \mathbb{Q}(\sqrt{D})$
$\alpha \in \mathcal{O}_K^+$ satisfies (M) if $\exists \delta \in \mathcal{O}_K^{\vee,+}$ such that

$$\mathrm{Tr}(\delta\alpha) = \min_{\beta \in \mathcal{O}_K^+} (\mathrm{Tr}(\delta\beta)).$$

- $\alpha \in \mathcal{O}_K^+$ satisfies $(M) \Rightarrow \alpha$ is a square (and indecomposable)
- Each totally positive unit is a square ($\Rightarrow$ have units of all signatures)
- $\alpha \in \mathcal{O}_K^+$ satisfies $(M) \Rightarrow \alpha$ is a unit

Quadratic fields 2

Theorem (K-Yatsyna 2019)

K = totally real number field of degree 2 that admits a universal $\mathbb{Z}$-form. Then $K = \mathbb{Q}(\sqrt{5})$.

Idea of proof

Assume Q is universal $\mathbb{Z}$-form over $K = \mathbb{Q}(\sqrt{D})$
Have units of all signatures: let $\varepsilon > 0$, $\varepsilon' < 0$
Case. $D \equiv 2, 3 \pmod 4$.

- $\delta := \frac{\varepsilon}{2\sqrt{D}} \in \mathcal{O}_K^+$, in fact $\mathcal{O}_K^\vee = \delta\mathcal{O}_K$
- $\alpha := \varepsilon^{-1} \cdot 2\sqrt{D}$ has $\mathrm{Tr}(\delta\alpha) = \mathrm{Tr}(1/2) = 1$, and so α satisfies (M)
- $\alpha \in \mathcal{O}_K^+$ satisfies $(M) \Rightarrow \alpha$ is a unit
- $4D = N(\alpha) = 1$ – contradiction!

Case. $D \equiv 1 \pmod 4$: similar

Higher degrees

Theorem (K-Yatsyna 2019)

K = totally real number field of degree d that admits a universal $\mathbb{Z}$-form.

- *If $d = 3, 4, 5, 7$ and K has principal codifferent ideal $\mathcal{O}_K^\vee$, then $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.*
- *$Q = X^2 + Y^2 + Z^2 + W^2 + XY + XZ + XW$ is universal over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$*

Idea of proof

Siegel's formula relates # elements of $\mathcal{O}_K^\vee$ of small trace to value of Dedekind zeta $\zeta_K(-1)$

$d = 3, 4, 5, 7 \Rightarrow$ only elements of trace 1 appear ($\Leftarrow$ dimensions of spaces of modular forms)

Estimate things $\Rightarrow$ only 1 candidate field in degrees $3, 4, 5$

$K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1}) \Rightarrow Q$ has class number one by mass formula

Thanks for your attention!

Slides at `sites.google.com/site/vitakala/`