Additively indecomposable integers in number fields

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Outline

Introduction

- K is a totally real number field with ring of integers $\mathcal{O}_K$
- embeddings $\sigma_1, \sigma_2, \ldots, \sigma_n : K \to \mathbb{R}$
- $\alpha \succ \beta$ if $\sigma_1(\alpha) > \sigma_1(\beta)$, $\ldots$, $\sigma_n(\alpha) > \sigma_n(\beta)$
- α is totally positive if $\alpha \succ 0$
- $\mathcal{O}_K^+$ is the semiring of all totally positive integers
- $\alpha \in \mathcal{O}_K^+$ is *(additively) indecomposable* if $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in \mathcal{O}_K^+$

Want to study structure of indecomposable elements
Motivation comes from universal quadratic forms
Will explain why interested in 24 009 857 226 825 282 345 490

Outline

1. Introduction
2. Universal quadratic forms
3. Indecomposables and continued fractions

Universal forms over $\mathbb{Z}$

A quadratic form with integral coefficients is *universal* if it represents all positive integers.
There are many indefinite forms, eg. $x^2 - y^2 - dz^2$ with $4 \nmid d$.
Consider positive definite *diagonal* forms

$$Q(x) = \sum_{1 \leq i \leq M} a_i x_i^2 = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_M x_M^2.$$

- No ternary universal positive forms
- Lagrange (1770): $x^2 + y^2 + z^2 + t^2$ is universal
- Ramanujan, Dickson (1916): classified quaternary universal positive forms, eg. $x^2 + 2y^2 + 4z^2 + dt^2$ with $d \leq 14$
- Conway-Schneeberger, Bhargava (1993): A positive diagonal form over $\mathbb{Z}$ is universal iff it represents $1, 2, 3, \ldots, 15$.

Universal forms over number fields

Study totally positive definite (diagonal) universal forms over totally real number fields, i.e., forms representing all totally positive integers.

- Siegel (1945): If K is totally real such that sum of M squares is universal, then $K = \mathbb{Q}$ ($M = 4$) or $K = \mathbb{Q}(\sqrt{5})$ ($M = 3$).
- Chan-Kim-Raghavan (1996): Determined all ternary universal forms over $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{5})$
- Kim (1999): 8-ary universal form over $\mathbb{Q}(\sqrt{n^2 - 1})$, but only finitely many $\mathbb{Q}(\sqrt{D})$ admit 7-ary universal forms.
- Blomer-K (2015): For each M, there are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{D})$ that do not admit universal M-ary forms.
- K-Svoboda (2016): For each M, there are infinitely many real biquadratic fields $\mathbb{Q}(\sqrt{D}, \sqrt{E})$ that do not admit universal M-ary forms.

Universal forms over number fields – proofs

Theorem (Blomer-K, 2015)

For each M, there are infinitely many real fields $\mathbb{Q}(\sqrt{D})$ that do not admit universal M-ary forms.

Idea of the proof

- $Q(x) = \sum_{1\le i\le M} a_i x_i^2$ with $a_i \in \mathcal{O}_K^+$
- Assume Q is universal, i.e., for all $\alpha \in \mathcal{O}_K^+$ there are $x_i \in \mathcal{O}_K$ such that $\alpha = Q(x)$.
- For α indecomposable have $\alpha = a_1x_1^2 + \cdots + a_Mx_M^2$, and so $\alpha = a_ix_i^2$
- Hence it suffices to choose $\mathbb{Q}(\sqrt{D})$ which has many (squarefree) indecomposables $\Rightarrow$ lower bound on number of variables M

With Svoboda extend to biquadratic $\mathbb{Q}(\sqrt{D}, \sqrt{E})$ by choosing large E so that indecomposables remain indecomposable

Outline

Lower bound

Want to understand structure of indecomposables, size of their norms, squarefreeness, etc.

Notation:

- $D > 0$ squarefree, $D \equiv 2, 3 \pmod 4$ (for simplicity)
- $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$
- the conjugate of $\alpha = x + y\sqrt{D}$ is $\alpha' = x - y\sqrt{D}$
- $N(\alpha) = x^2 - y^2 D$
- $\alpha \succ \beta$ iff $\alpha > \beta$ and $\alpha' > \beta'$

Proposition

If $\alpha \in \mathcal{O}_K^+$ satisfies $N(\alpha) < 2\sqrt{D}$ and $n \nmid \alpha$ for all $n \geq 2$, then α is indecomposable.

In particular, units are indecomposable.

Continued fractions

- The continued fraction

$$\sqrt{D} = [u_0, \overline{u_1, u_2, \dots, u_{s-1}, 2u_0}] = u_0 + \frac{1}{u_1 + \frac{1}{u_2 + \cdots}}$$

  is periodic and the sequence $(u_1, u_2, \dots, u_{s-1})$ is symmetric.
- $\frac{p_i}{q_i} := [u_0, u_1, \dots, u_i] = i$th convergent
- $\alpha_i := p_i + q_i\sqrt{D}$, $\alpha_{-1} := 1$
- $\alpha_{i+1} = u_{i+1}\alpha_i + \alpha_{i-1}$
- $\alpha_i \succ 0$ iff i is odd
- α_{s-1} is the fundamental unit
- $N(\alpha_i) \approx \frac{2\sqrt{D}}{u_{i+1}}$ is small

Suggests a connection between convergents α_i and indecomposables!

Characterization of indecomposables

Proposition (classical)

The indecomposable integers of $\mathbb{Z}[\sqrt{D}]$ are exactly the "semiconvergents"

$$\alpha_{i,r} := \alpha_i + r\alpha_{i+1}$$

and $\alpha'_{i,r}$ for odd $i \geq -1$ and $0 \leq r < u_{i+2}$.

Remark

Number of indecomposables is the sum S of odd coefficients of continued fraction. Seems to be (roughly) the minimal number M of variables needed by a (diagonal) universal form over $\mathbb{Q}(\sqrt{D})$.
With Blomer:

$$\frac{S}{2s} \leq M \leq 8S$$

Upper bound

There are finitely many indecomposables upto multiplication by units, and so there is an upper bound on their norms:

Theorem (Dress-Scharlau, 1982)

If α is indecomposable, then $N(\alpha) \le D$.

What is the actual maximum? The upper bound can be improved:

Theorem (Jang-Kim, 2016)

$$N(\alpha_{i,r}) = \frac{D - (T_{i+1} + rN(\alpha_{i+1}))^2}{|N(\alpha_{i+1})|} \le \frac{D}{N},$$

$$N := \min\left\{|N(\alpha)|, \alpha \in \mathbb{Z}[\sqrt{D}] \text{ s.t. } N(\alpha) < 0\right\} = \min\{|N(\alpha_i)|, i \text{ even}\}$$

is the minimum of absolute values of negative norms, and $T_{i+1} \in \mathbb{Z}$.

Perhaps the maximal norm is of the form $\frac{D-a^2}{N}$ for some a?

Conjecture

Conjecture (Jang-Kim, 2016)

Let a be the smallest nonnegative rational integer such that N divides $D - a^2$. Then $N(\alpha) \leq \frac{D-a^2}{N}$ for all indecomposable $\alpha \in \mathbb{Z}[\sqrt{D}]$.

Recall

$$N(\alpha_{i,r}) = \frac{D - *^2}{|N(\alpha_{i+1})|}, N := \min\left\{|N(\alpha_{i+1})|, i+1 \text{ even}\right\}.$$

Conjecture is based on the expectation that $N(\alpha_{i,r})$ is maximal when $|N(\alpha_{i+1})|$ is minimal.
Is it true? Want to estimate sizes of norms in terms of continued fraction coefficients u_i.

Norms of convergents

Recall $\sqrt{D} = [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, 2u_0}]$, $\frac{p_i}{q_i} = [u_0, u_1, \ldots, u_i]$
$\alpha_i = p_i + q_i\sqrt{D}$, $N_i := |N(\alpha_i)|$
Define $c_i := [u_i, u_{i+1}, u_{i+2}, \ldots]$, $c_i = u_i + \frac{1}{c_{i+1}}$.
$\sqrt{D} = \frac{c_{i+1}p_i + p_{i-1}}{c_{i+1}q_i + q_{i-1}}$, which implies

$$N_i = \frac{2\sqrt{D}}{c_{i+1}} - \frac{N_{i-1}}{c_{i+1}^2}.$$

Can use this recursively to express N_i as a power series in c_j^{-1} and u_j^{-1}.

Proposition

$$\frac{N_i}{2\sqrt{D}} = \frac{1}{u_{i+1}}\left(1 - \frac{1}{u_i u_{i+1}} - \frac{1}{u_{i+1}u_{i+2}}\right) + \cdots$$

Norms of indecomposables

Indecomposables are $\alpha_{i,r} := \alpha_i + r\alpha_{i+1}$ for odd $i \geq -1$ and $0 \leq r < u_{i+2}$, have

$$N(\alpha_{i,r}) = \frac{D - (T_{i+1} + rN(\alpha_{i+1}))^2}{|N(\alpha_{i+1})|}.$$

Quadratic in r, compute that for fixed i, $N(\alpha_{i,r})$ is maximal when $r = u_{i+2}/2$ (if u_{i+2} even).
Then

Proposition

$$\frac{2N(\alpha_{i,r})}{\sqrt{D}} = u_{i+2} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} + \cdots$$

Construction

Want to choose $\sqrt{D} = [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, 2u_0}]$ with suitable continued fraction s.t. for some odd $i < j$

1. α_{i+1} is the element with the largest negative norm ($=$ the smallest norm in absolute value $N_{i+1} = N$)
2. $N_{i+1} = |N(\alpha_{i+1})| < N_{j+1} = |N(\alpha_{j+1})|$
3. $N(\alpha_{i,r}) < N(\alpha_{j,t})$

By Friesen's theorem, can prescribe coefficients $u_1, u_2, \ldots, u_{s-1}$.

$$\frac{N_{i+1}}{2\sqrt{D}} = \frac{1}{u_{i+2}}\left(1 - \frac{1}{u_{i+1}u_{i+2}} - \frac{1}{u_{i+2}u_{i+3}}\right) + \cdots$$

$$\frac{2N(\alpha_{i,r})}{\sqrt{D}} = u_{i+2} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} + \cdots$$

(1) $\Rightarrow$ s even (else fundamental unit has norm -1)
(2) "$\Rightarrow$" $u_{i+2} \geq u_{j+2}$
(3) "$\Rightarrow$" $u_{i+2} \leq u_{j+2}$

Construction continued

Recall $N_{i+1} < N_{j+1}$, $N(\alpha_{i,r}) < N(\alpha_{j,t})$

$$\frac{N_{i+1}}{2\sqrt{D}} = \frac{1}{u_{i+2}}\left(1 - \frac{1}{u_{i+1}u_{i+2}} - \frac{1}{u_{i+2}u_{i+3}}\right) + \cdots$$

$$\frac{2N(\alpha_{i,r})}{\sqrt{D}} = u_{i+2} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} + \cdots$$

Thus $u_{i+2} = u_{j+2}$.

Then we get $\frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} \leq \frac{1}{u_{j+1}} + \frac{1}{u_{j+3}}$ and $\frac{1}{u_{i+1}} + \frac{1}{u_{i+3}} \geq \frac{1}{u_{j+1}} + \frac{1}{u_{j+3}} \Rightarrow$ again equality.

Need to go one step further in power series.

Counterexample

Theorem (K)

Let $D = 24\,009\,857\,226\,825\,282\,345\,490$*. Then:*

1. $D \equiv 2 \pmod 4$ *is squarefree and its continued fraction is*

$$\sqrt{D} = [u_0, \overline{10, 2, 12, 6, 1, 3, 4, 3, 12, 3, 4, 2, 1, 6, 12, 2, 10, 2u_0}]$$

   with $u_0 = 154\,951\,144\,645$*.*

2. α_2 *has the largest negative norm* $-N = -24\,548\,583\,881$
3. $\alpha_{7,6}$ *is the indecomposable integer with the largest norm* $977\,608\,342\,706$
4. *The smallest nonnegative rational integer* a *such that* N *divides* $D - a^2$ *is* $a = 4\,030\,160\,489$*.*
5. $N(\alpha_{7,6}) = 977\,608\,342\,706 > \frac{D-a^2}{N} = 977\,393\,040\,249 = N(\alpha_{1,6})$

Hence the conjecture is false over $\mathbb{Q}(\sqrt{D})$*.*

Thanks

Thanks for your attention!

Slides at `sites.google.com/site/vitakala/`