INFINITY IS RELEVANT

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CSP (Constraint Satisfaction Problem)

- satisfiability of a list of constraints
  - $(x, y, z) \in R$
    $(x, x, u) \in S$
    $u \in T$
  - R, S, T - relations on a domain D
  - x, y, z, u - variables in X
  - $f : X \to D$ is a solution if it satisfies all the constraints
  - $(f(x), f(y), f(z)) \in R$
    $(f(x), f(x), f(u)) \in S$
    $f(u) \in T$
  - Decision: Does a solution exist?
  - Search: Find a solution.
- $\mathcal{D}$ - finite set of relations on a domain D
  $CSP(\mathcal{D})$ - restriction of CSP to instances in which the domain is D and all constraint relations are from $\mathcal{D}$

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CSP

- homomorphism problem
  - $$\left.\begin{array}{l}\mathcal{E} = (E; S_1, S_2, \ldots, S_n) \\ \mathcal{D} = (D; R_1, R_2, \ldots, R_n)\end{array}\right\} \text{similar relational structures}$$
  - $h : E \to D$ is a homomorphism from $\mathcal{E}$ to $\mathcal{D}$ if $(a_1, a_2, \ldots, a_k) \in S_i \Rightarrow (h(a_1), h(a_2), \ldots, h(a_k)) \in R_i$
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Example - 3-SAT

- example: $\varphi = (x \vee y \vee z) \wedge (\neg z \vee u \vee v) \wedge (\neg u \vee x \vee y)$
- list of constraints
  $(x, y, z) \in S_{000}$, $S_{000} = \{0,1\}^3 \backslash \{(0,0,0)\}$
  $(z, u, v) \in S_{100}$, $S_{100} = \{0,1\}^3 \backslash \{(1,0,0)\}$
  $(u, x, y) \in S_{100}$
- 3-SAT is equivalent to $CSP(\mathcal{D}_{3SAT})$ with $D_{3SAT} = \{0,1\}$ and $\mathcal{D}_{3SAT} = \{S_{ijk} : i, j, k \in \{0,1\}\}$, where $S_{ijk} = \{0,1\}^3 \backslash \{(i,j,k)\}$
- homomorphism problem:
  $E = \{x, y, z, u, v\}$
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Examples

- 3-SAT (NP-complete)
- 1-in-3-SAT (NP-complete)
  1-in-3 $= (\{0,1\}; \{(1,0,0),(0,1,0),(0,0,1)\})$
- NAE-3-SAT (NP-complete)
  NAE-3 $= (\{0,1\}; \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\})$
- 3-coloring of a graph (NP-complete)
  $(\{0,1,2\}; \neq)$
- 2-coloring of a graph (P)
  $(\{0,1\}; \neq)$
- system of linear equations over $\mathbb{Z}_5$ (P)

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Selected results

- $CSP(\mathcal{A})$, $|A| = 2$, is in P or NP-complete [Schaefer '78]
- H - undirected graph
  H bipartite $\Rightarrow$ H-coloring $\in P$
  otherwise $\Rightarrow$ H-coloring is NP-complete [Hell, Nešetřil '90]
- $CSP(\mathcal{A})$, A - finite, is in P or NP-complete [Bulatov '17];[Zhuk '17]

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Examples of CSPs over infinite templates

- $(\mathbb{Z}; \{(x, y, z) : x + y + z = 1\})$ (P)
- $(\mathbb{Q} \backslash \{\frac{1}{3}\}; \{(x, y, z) : x + y + z = 1\})$ (P)

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PCSP (Promise CSP)

- $PCSP(\mathcal{A}, \mathcal{B})$
- $\mathcal{A}, \mathcal{B}$ - relational structures, $\mathcal{A} \to \mathcal{B}$
- Search: Given $\mathcal{X}$ such that $\mathcal{X} \to \mathcal{A}$ find $\mathcal{X} \to \mathcal{B}$.
- $PCSP(\mathcal{A}, \mathcal{A}) = CSP(\mathcal{A})$
- $(\mathcal{A}, \mathcal{B}), (\mathcal{A}', \mathcal{B}')$ - PCSP templates
  $\mathcal{A}' \to \mathcal{A}, \mathcal{B} \to \mathcal{B}' \Rightarrow PCSP(\mathcal{A}', \mathcal{B}') \leq PCSP(\mathcal{A}, \mathcal{B})$
- examples:
  - 6-coloring of a 3-colorable graph
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$PCSP$(1-in-3,NAE-3)

- find a valid NAE-3-SAT assignment to a 1-in-3 satisfiable instance
- P
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$$\left.\begin{aligned}(x,y,z) &\rightsquigarrow x+y+z=1\\(u,y,x) &\rightsquigarrow u+y+x=1\end{aligned}\right\}\text{ over }\mathbb{Z}$$

  - The obtained system is solvable.
  - Finding a solution to a system of linear equations over $\mathbb{Z}$ is in P. [Grötschel, Lovász, Schrijver '93]
  - ϕ - solution to the system
  - $\psi(x) = \begin{cases}0, & \phi(x) \le 0\\ 1, & \phi(x) > 0\end{cases}$ - a valid NAE-3-SAT assignment

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- alternatively, solve the system over $\mathbb{Q}\backslash\{\frac{1}{3}\}$
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Main theorem

Theorem (Barto)

Let $\mathcal{C}$ be a finite relational structure such that 1-in-3$\rightarrow \mathcal{C}$ and $\mathcal{C} \rightarrow$NAE-3. Then $CSP(\mathcal{C})$ is NP-complete.

Preliminaries

Definition

Let $\mathcal{C}$ be a CSP template. $s : C^n \to C$ is a *polymorphism* of $\mathcal{C}$ if for each relation R in $\mathcal{C}$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} \in R, \dots, \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in R \Rightarrow \begin{bmatrix} s(a_{11}, \dots, a_{1n}) \\ \vdots \\ s(a_{m1}, \dots, a_{mn}) \end{bmatrix} \in R.$$

Definition

$s : C^n \to C$ is cyclic if

$$s(a_1, a_2, \dots, a_n) = s(a_2, \dots, a_n, a_1)$$

Theorem (Barto, Kozik '12)

Let $\mathcal{C}$ be a finite CSP template. If $CSP(\mathcal{C})$ is not NP-complete, then $\mathcal{C}$ has a cyclic polymorphism of arity p for every prime number $p > |C|$.

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Infinity is necessary

Theorem

Let $\mathcal{C} = (C; R)$ be a finite relational structure with ternary $R \subseteq C^3$ such that 1-in-3$\rightarrow \mathcal{C}$ and $\mathcal{C} \rightarrow$NAE-3. Then $CSP(\mathcal{C})$ is NP-complete.

Sketch of the proof

- assume $CSP(\mathcal{C})$ is not NP-complete
- $g : \mathcal{C} \to$NAE-3
- s - cyclic polymorphism of prime arity $p > 60|C|$
- $t(x_{11}, x_{12}, \ldots, x_{1p}, x_{21}, x_{22}, \ldots, x_{2p}, \ldots, x_{p1}, x_{p2}, \ldots, x_{pp}) = s(s(x_{11}, x_{21} \ldots, x_{p1}), s(x_{12}, x_{22}, \ldots, x_{p2}) \ldots, s(x_{1p}, x_{2p}, \ldots, x_{pp}))$
- $$t \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \ldots & x_{pp} \end{pmatrix}$$

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Definition

Let $X = (x_{ij})$, Y be $p \times p$ zero-one matrices. The *area* of X is

$$\lambda(X) = \left(\sum_{i,j} x_{ij} \right) / p^2.$$

X and Y are *g-equivalent*, denoted $X \sim Y$, if $g(t(X)) = g(t(Y))$.
X is *tame* if

$$X \sim 0_{p\times p} \text{ if } \lambda(X) < 1/3$$
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Resuming the proof sketch

- show that t is cyclic
- show $0_{p\times p} \not\sim 1_{p\times p}$

Definition

Let $1 \leq k_1, \dots, k_p \leq p$. By $[k_1, \dots, k_p]$ we denote the matrix whose i-th column begins with k_i ones followed by $(p - k_i)$ zeroes.
An *almost rectangle* is $[k, \dots, k, l, \dots, l]$ where $0 \leq k - l \leq 5|C|$.

- show that almost rectangles are tame
- construct two almost rectangles X_1 and X_2 such that $\lambda(X_1) < 1/3$ and $\lambda(X_2) > 1/3$, but $t(X_1) = t(X_2)$
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- $X_1 \not\sim X_2$
  Contradiction!

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Thanks for your attention!