Trivial connections on discrete surfaces

Jana Vráblíková

April 2019

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- Design tangent vector field on a discrete surface that is as smooth as possible

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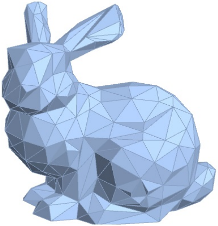

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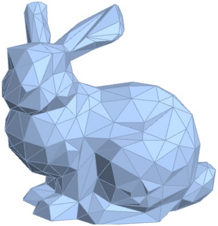

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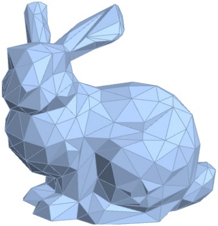

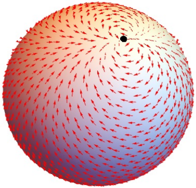

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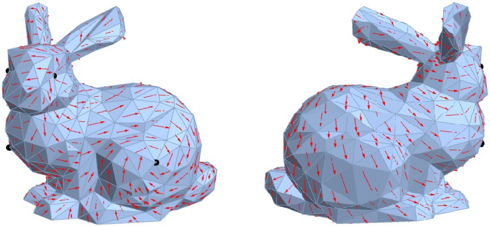

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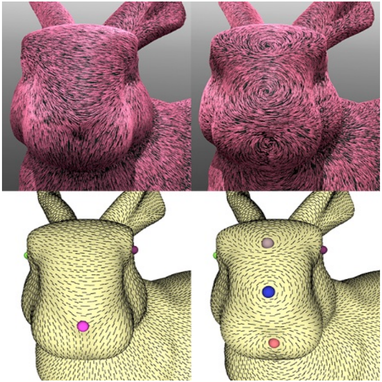

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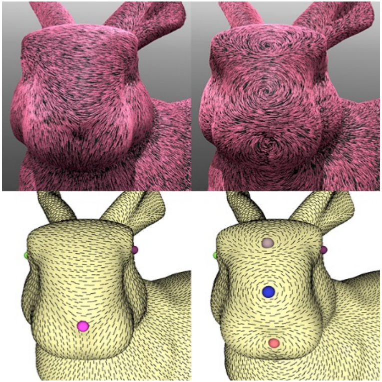

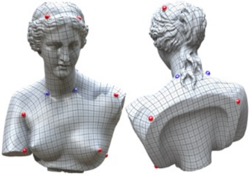

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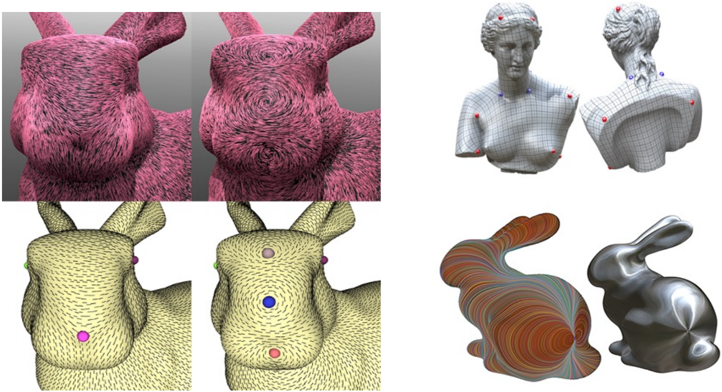

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Differential geometry - basic notions

- compact surface S

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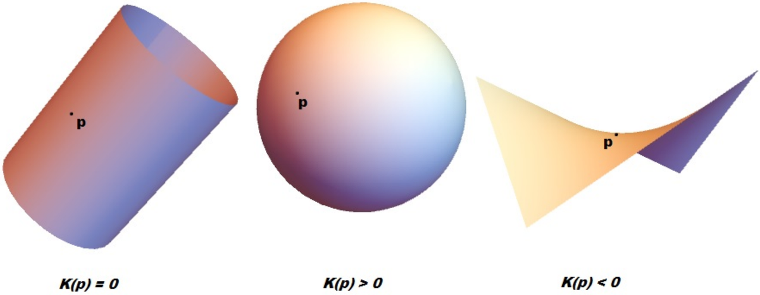

Differential geometry - basic notions

- compact surface S
- Gaussian curvature K

Theorem (Gauss-Bonnet theorem)

Let S be compact oriented surface, then

$$\int_S K \, d\mathcal{H}^2 = 2\pi(2 - 2g),$$

where g is genus of the surface S.

Discrete differential geometry

Definition (Triangle mesh)

Triangle mesh in $\mathbb{R}^3$ is a triple $S = (V, E, F)$, where $\forall v \in V : v \in \mathbb{R}^3$, $E \subseteq V \times V$ and $F \subseteq V \times V \times V$.

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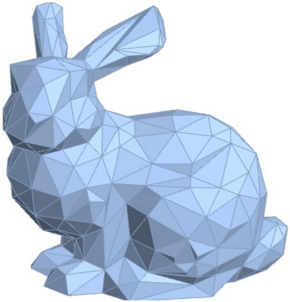

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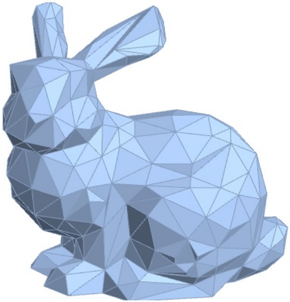

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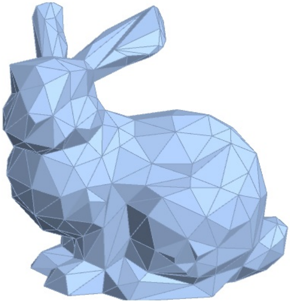
- without boundary



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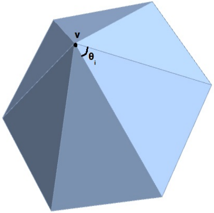
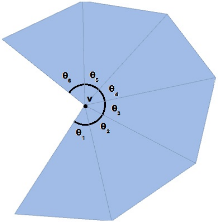
Definition (Discrete Gaussian curvature)

Let $S = (V, E, F)$ be triangle mesh. For vertex $v \in V$ we define its *discrete Gaussian curvature* as 2π - $\sum_i \theta_i$, where θ_i are angles in adjacent faces to the vertex v





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Theorem (Discrete Gauss-Bonnet theorem)

Let $S = (V, E, F)$ be compact triangle mesh. Then

$$\sum_{v \in V} K_v = 2\pi\chi,$$

where $\chi = |V| - |E| + |F| = 2 - 2g$, is Euler characteristic of the surface and g is genus of S.

Dual mesh

Definition (Dual mesh)

Let $S = (V, E, F)$ be triangle mesh. We define its *dual mesh* S' as triple $S' = (V', E', F')$, where

- vertices $v' \in V'$ lie in centroid of faces F
- faces $f' \in F'$ are polygons with vertices in V'
- edge $(v_i', v_j') \in E'$ iff faces $f_i, f_j \in F$ shared an edge

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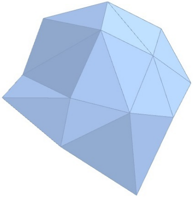

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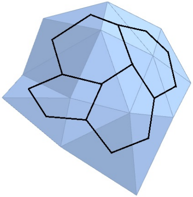

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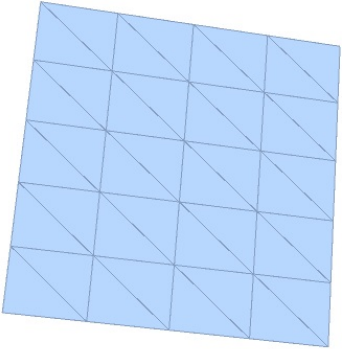
Let $S' = (V, E, F)$ be a dual mesh of a triangle mesh S. *Discrete connection* $\omega : E' \rightarrow \mathbb{R}$, that each dual edge e' assign an angle.

Definition (Levi-Civita connection)

Let $\Phi : E' \rightarrow \mathbb{R}$ be discrete connection. If $\forall e' \in E' : \omega(e') = 0$ we call ω *a discrete Levi − Civita connection*.

Levi-Civita connection

- plane



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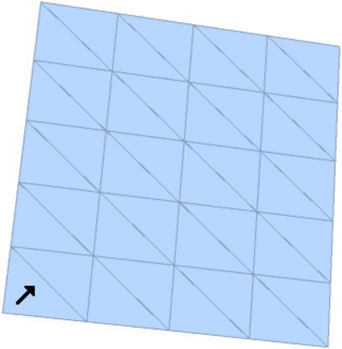

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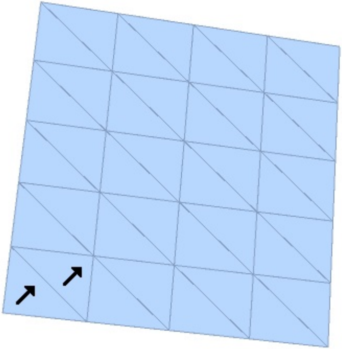

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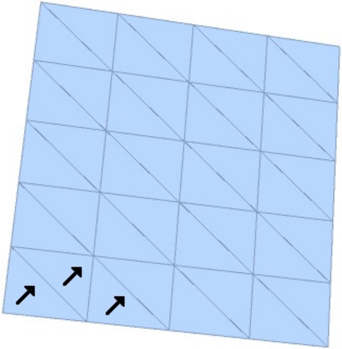

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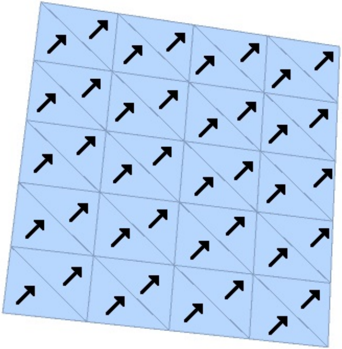

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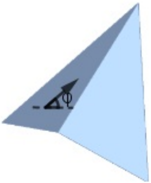

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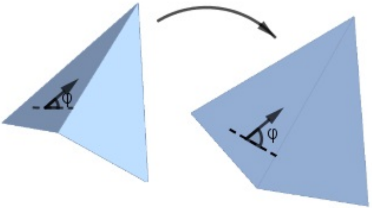

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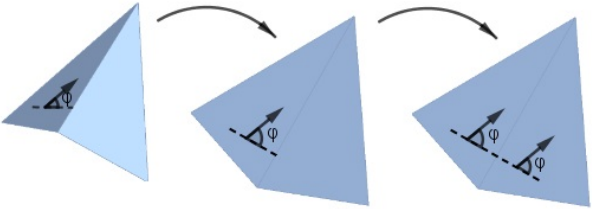

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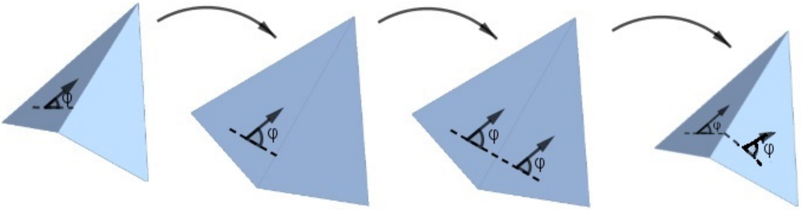

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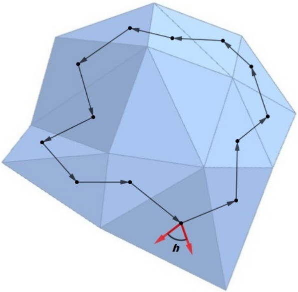

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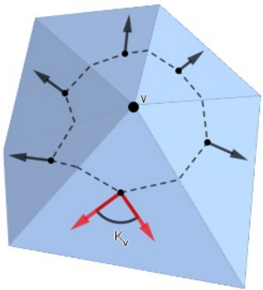

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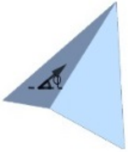

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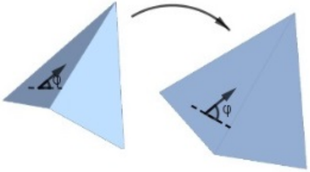

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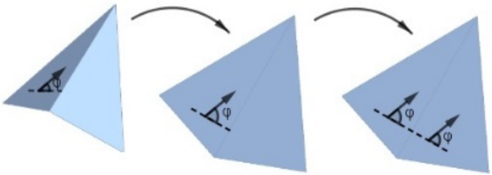

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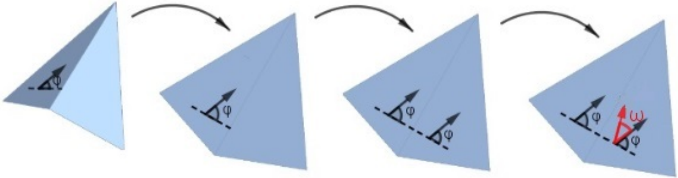

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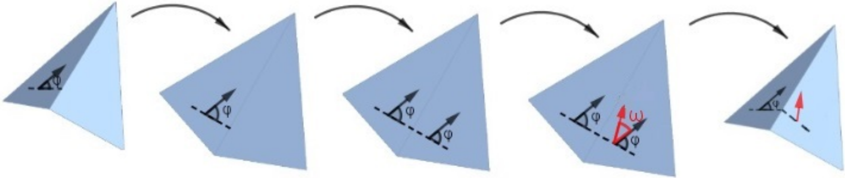

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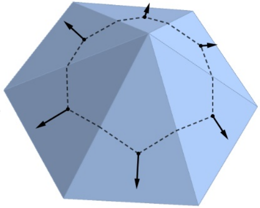

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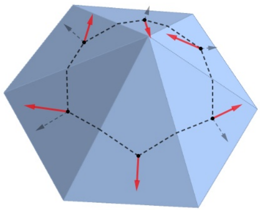

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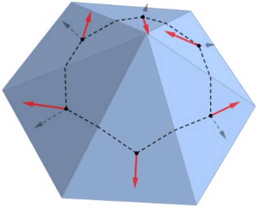

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Theorem (Poincaré index theorem)

For any surface $S = (V, E, F)$

$$\sum_{v \in V} index_v = \chi,$$

where χ is Euler characteristic of S.

Basis

- zero holonomy around every cycle

Basis

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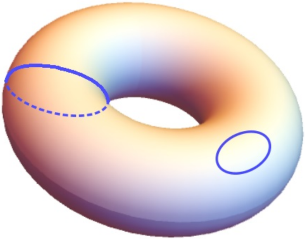
- find a basis of cycles on the surface

Basis

- zero holonomy around every cycle
- find a basis of cycles on the surface
- contractible and non-contractible loops



Linear system

- for each base loop formed by dual edges $e'_1, e'_2, \ldots, e'_l$ we have one linear equation

$$\omega(e'_1) + \omega(e'_2) + \cdots + \omega(e'_l) = h,$$

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- for $2g$ non-contractible generators we can form matrix $\mathbf{H} \in \{0,1\}^{2g\times|E|}$
- linear system $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{pmatrix} \mathbf{D}^T \\ \mathbf{H}^T \end{pmatrix}$, where $\mathbf{A} \in \{0,1\}^{|E|\times(|V|+2g)}$ and $\vec{b}$ is a vector of holonomies

Linear system

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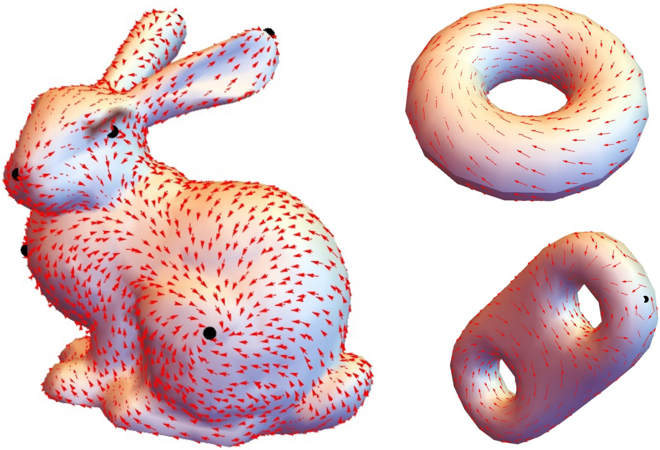
$$\min_{x \in \mathbb{R}^{|E|}} ||\mathbf{A}\vec{x} - \vec{b}||$$

Results



Thank you for your attention!