Number systems in the complex plane and in lattices

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Seč

About me & this presentation

- Currently: 1st year of PhD in Number theory (under Víťa Kala)
- Today's talk: Nontraditional number systems
  - My former topic (as part of research group TIGR at FNSPE CTU ("Jaderka")
  - The last part – joint results with A. Kovács, Budapest
  - It's going to be easy: Feel free to ask!

Outline

1. GNSs in the complex plane
2. GNSs in lattices
3. Infinitely many GNSs with the same radix

Number systems in $\mathbb{Z}$

Examples of number systems in $\mathbb{Z}$:

- The binary system: radix 2, alphabet $\{0, 1\}$. Represents all nonnegative integers. Not a GNS.
- The decimal system: radix 10, alphabet $\{0, 1, \dots, 9\}$. Represents all nonnegative integers. Not a GNS.

Definition

Having a nonzero **radix** $\beta \in \mathbb{Z}$ and a finite **alphabet** $\mathcal{A} \subset \mathbb{Z}$ containing 0: The pair $(\beta, \mathcal{A})$ is a **GNS** in $\mathbb{Z}$ if every element $0 \neq x \in \mathbb{Z}$ has a unique representation of the form

$$x = \sum_{k=0}^{N} a_k \beta^k, \qquad N \in \mathbb{N}_0,\ a_k \in \mathcal{A},\ a_N \neq 0.$$

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Number systems in $\mathbb{Z}$

Further examples of number systems in $\mathbb{Z}$:

- Negabinary system (Vittorio Grünwald, 1885): Radix -2, alphabet $\{0, 1\}$. A GNS in $\mathbb{Z}$.
- Weighted ternary system: Radix 3, alphabet $\{-1, 0, 1\}$. A GNS in $\mathbb{Z}$.

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A peek into other rings

$\mathbb{Z}$ can be replaced by any ring R. For us mostly $R \subset \mathbb{C}$, discrete.
The main example: Gaussian integers $\mathbb{Z}[\mathrm{i}] = \{a + b\mathrm{i} \mid a, b \in \mathbb{Z}\}$.

Definition

Having a nonzero **radix** $\beta \in R$ and a finite **alphabet** $\mathcal{A} \subset R$ containing 0: The pair $(\beta, \mathcal{A})$ is a **GNS** in R if every element $0 \neq x \in R$ has a unique representation of the form

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- The system $(-2, \{0, 1, \mathrm{i}, 1 + \mathrm{i}\})$ is a GNS in $\mathbb{Z}[\mathrm{i}]$.
- Penney, 1965: $(-1 + \mathrm{i}, \{0, 1\})$ is a GNS in $\mathbb{Z}[\mathrm{i}]$.
  - $-1 = (11101)_{-1+\mathrm{i}} = (-1+\mathrm{i})^4 + (-1+\mathrm{i})^3 + (-1+\mathrm{i})^2 + (-1+\mathrm{i})^0$.
- However, $(+1 + \mathrm{i}, \{0, 1\})$ is not a GNS in $\mathbb{Z}[\mathrm{i}]$.

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Necessary conditions for a GNS

Proposition

If $(\beta, \mathcal{A})$ is a GNS in R, then:

- $\mathcal{A}$ is a complete residue system modulo β,
- $|\beta| \neq 0, 1$,
- $|1 - \beta| \neq 1$.

These conditions are not sufficient! The decimal system satisfies them all.

Algorithm for finding the representation of $x \in R$

(Supposing $\mathcal{A}$ is a CRS modulo β.)

- Find the last digit a (by congruence);
- compute the **successor** $\varphi(x) := (x - a)\beta^{-1}$;
- find the representation of $\varphi(x)$.

Example: Compute the representation of 1 in $(3, \{-11, 0, 5\})$.

- $x = 1$.
- $x = 1 \equiv \mathbf{-11} \pmod 3$. $\varphi(x) = (1 - (-11))/3 = 4$.

Example 2: In $(-1 + \mathrm{i}, \{0, 1\})$, we have $-1 = (11101)_{-1+\mathrm{i}}$.

Example 3: There is no representation of -16 in the decimal system.

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- We have found the representation $1 = (5(-11)(-11))_3 = 5 \cdot 3^2 + (-11) \cdot 3 + (-11)$.

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Corollaries of the algorithm:

- If $\mathcal{A}$ is a CRS modulo β, then every $x \in R$ has at most one representation.
- The system $(\beta, \mathcal{A})$ is a GNS iff the algorithm terminates for every $x \in R$.

Proposition

If $\mathcal{A}$ is a complete residue system modulo β and $|\beta| \neq 0, 1$, then $(\beta, \mathcal{A})$ is a GNS if and only if there exist a representation of all elements of the **testing set**

$$T := \left\{ x \in R : |x| \leq \frac{K}{|\beta| - 1} \right\}, \qquad \text{where } K := \max_{a \in \mathcal{A}} |a|.$$

- Idea of the proof: If $x \notin T$, then $|\varphi(x)| < |x|$.
- Furthermore: If $x \in T$, then $\varphi(x) \in T$. Therefore all periodic points of φ lie inside of T.

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Crutial note: If R is discrete, then it is simple to check the GNS property for any given $(\beta, \mathcal{A})$.

Questions:

1. Fully characterise GNSs with a given radix.
2. Examine a given type of alphabet for all radices.
3. Characterise all possible radices in the given ring.
4. Many other questions of various flavours: algorithmization, topology, dynamical systems, ...

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    - Open even for $R = \mathbb{Z}$, $\beta = 3$.
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    - $\{0, 1, \ldots, |\beta| - 1\}$ gives a GNS in $\mathbb{Z}$ iff $\beta \leq -2$.
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Answers in $\mathbb{Z}[\mathrm{i}]$:

1. Fully characterise GNSs with a given radix.
   - Hopeless.
2. Examine a given type of alphabet for all radices.

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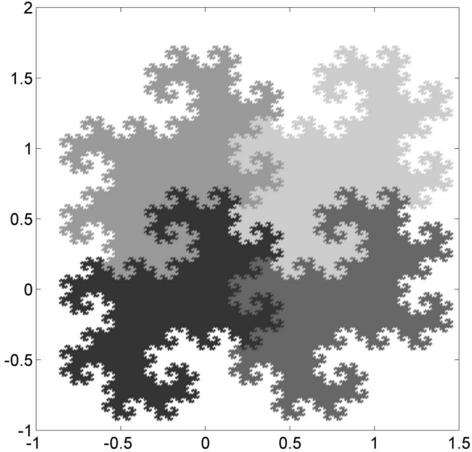

Figure: Tiling of the complex plane generated by the Penney system $(-1+\mathrm{i}, \{0,1\})$.

A more general setting: lattices

- A **lattice** ($\mathbb{Z}^d$);
- a **radix** $L \in \mathbb{Z}^{d \times d}$;
- a finite **alphabet** $\mathcal{A} \subset \mathbb{Z}^d$ containing zero.

Definition

The pair $(L, \mathcal{A})$ is a **GNS** in $\mathbb{Z}^d$ if every nonzero element $x \in \mathbb{Z}^d$ has a unique representation of the form

$$x = \sum_{k=0}^{N} L^k a_k, \qquad N \in \mathbb{N}_0,\ a_k \in \mathcal{A},\ a_N \neq 0.$$

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Necessary conditions

- The alphabet $\mathcal{A}$ must be a complete residue system modulo L;
- the radix L must be expansive, i.e. $\rho(L^{-1}) < 1$ (Vince, 1993);
- $\det(I - L) \neq \pm 1$ (the “unit condition”).

Proposition

Suppose $L \in \mathbb{Z}^{d\times d}$ is expansive and $\mathcal{A}$ is a CRS modulo L. Take any vector norm satisfying $r := \left\|L^{-1}\right\|_* < 1$ and denote $K := \max_{d\in\mathcal{A}} \|d\|_*$. Then the pair $(L, \mathcal{A})$ is a GNS in $\mathbb{Z}^d$ if and only if there exists a representation for every element of the *testing set*

$$T := \left\{x \in \mathbb{Z}^d : \|x\|_* \leq \frac{Kr}{1-r}\right\}.$$

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The used digits lie in a parallelogram around the origin.

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If $\rho(L^{-1}) < 1/2$, then there always exists an alphabet such that $(L, \mathcal{A})$ is a GNS.

They use the **dense** alphabet, i.e. the smallest representative (in a certain vector norm) from every congruence class.

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- Radix is not expansive or the unit condition $\det(I - L) \neq \pm 1$ fails $\implies$ there is no such alphabet.
- For -2 in $\mathbb{Z}$, only the alphabets $\{0, 1\}$ and $\{0, -1\}$ are suitable.
- Similarly, for $-1 + i$ in $\mathbb{Z}[i]$ there are only four good alphabets.
- Matula, 1982: In $\mathbb{Z}$, for every β with $|\beta| \geq 3$ there are infinitely many alphabets.

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Sparse alphabets

Question 2: Can all the digits be far away from the origin?

Definition

Given a radix $L \in \mathbb{Z}^{d \times d}$, a sequence of alphabets $(\mathcal{A}_n)_{n \in \mathbb{N}}$ is called a **family of arbitrarily sparse alphabets** if for any given ball B around the origin, there exists an n_B such that for $n \geq n_B$ we have $\mathcal{A}_n \cap B = \{0\}$.

Equivalently we can require that for any finite $0 \notin S \subset \mathbb{Z}^d$, the alphabets $\mathcal{A}_n$ do not use any digits from S for $n \geq n_S$.

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I.e.: problems arise only if all factors of the characteristic polynomial have degree less than 3.

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Thank you for your attention.

Especially for any questions!

Starting point:

Theorem (Germán, Kovács, 2007)

If $\rho(L^{-1}) < 1/2$, then a GNS always exists.

Results:

Theorem

Suppose that $\rho(L^{-1}) \leq 1/2$ and 2 is not an eigenvalue of L. There always exist infinitely many GNSs with radix L except for the case where $d = 2$ and L has complex eigenvalues (where we do not know), and the case of radix -2 in $\mathbb{Z}$, where only two GNSs exist.

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