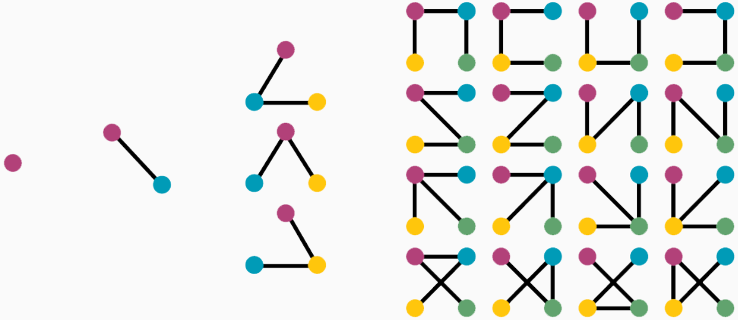
another proof of cayley's formula

an introduction to analytic combinatorics

Eva Hainzl, 2019

Cayley's formula

For every $n \in \mathbb{N}$, the number of labelled trees on n vertices is n^{n-2}.

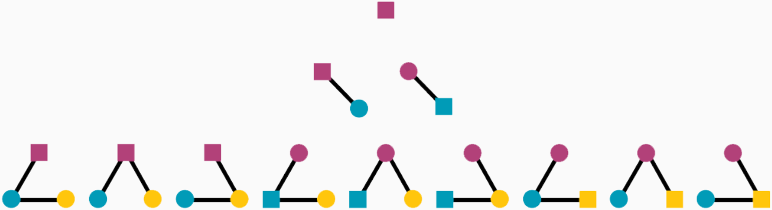


Cayley's formula

For every $n \in \mathbb{N}$, the number of labelled trees on n vertices is n^{n-2}.

Equivalent problem

For every $n \in \mathbb{N}$, the number of labelled, rooted trees on n vertices is n^{n-1}.



Overview

1. Find the generating function (implicit expression)
   Symbolic method
2. Analysis of the generating function
   Lagrange inversion theorem

Definitions

Definition

A combinatorial class, or simply a class, is a finite or denumerable set on which a size function is defined, satisfying the following conditions:

1. the size of an element is a non-negative integer
2. the number of elements of any given size is finite

Definitions

Definition

The counting sequence of a class $\mathcal{A}$ is the sequence of integers $(\mathrm{A_n})_{\mathrm{n} \geq 0}$ where $\mathrm{A_n}$ is the number of objects in class $\mathcal{A}$ that have size n.

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The ordinary generating function (OGF) of a combinatorial class $\mathcal{A}$ is the formal power series

$$\mathrm{A(z)} = \sum_{\mathrm{n}=0}^{\infty} \mathrm{A_n z^n}$$

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Definition

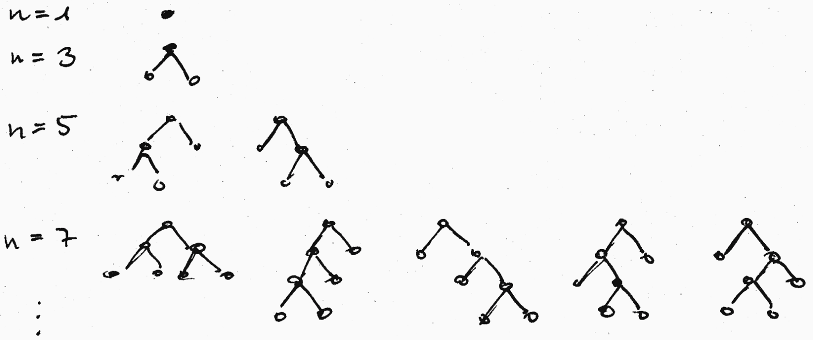
The exponential generating function (EGF) of a combinatorial class $\mathcal{A}$ is the formal power series

$$\mathrm{A(z)} = \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{A_n}}{\mathrm{n!}} \mathrm{z^n} \quad \mathrm{A_n} = \mathrm{n!} \cdot [\mathrm{z^n}]\mathrm{A(z)}$$

The Symbolic Method

The Symbolic Method

Example (Binary Trees (rooted, plane))



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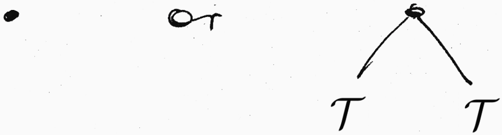

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- Atomic class: $\mathcal{Z}$ ={$\bullet$}, size = 1

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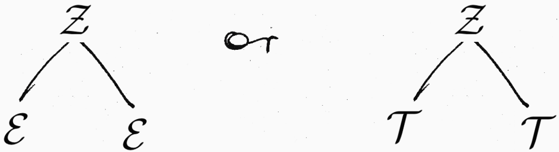
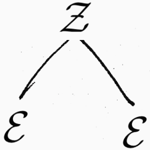
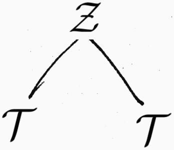


- Class of binary trees: $\mathcal{T}$ ={binary trees}, size = # of vertices
- Atomic class: $\mathcal{Z}$ ={$\bullet$}, size = 1
- Neutral class: $\mathcal{E}$ ={empty graph}, size = 0

Example (Binary Trees (rooted, plane))



or



$$\mathcal{T} = \mathcal{Z} \times \mathcal{E} \times \mathcal{E} + \mathcal{Z} \times \mathcal{T} \times \mathcal{T}$$

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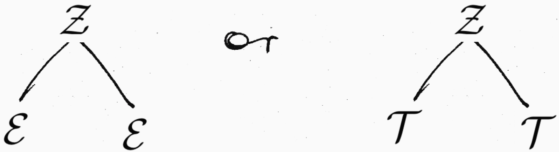

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$$\mathrm{T}(\mathrm{z}) = \mathrm{z} \cdot 1 \cdot 1 + \mathrm{z} \cdot \mathrm{T}(\mathrm{z}) \cdot \mathrm{T}(\mathrm{z}) = \mathrm{z}(1 + \mathrm{T}(\mathrm{z})^2)$$

The Symbolic Method

Observation

1. The EGF of $\mathcal{E}$ is 1.

   $\mathcal{E}$ contains by definition only a single object of size 0.

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The Symbolic Method

Observation

1. The EGF of $\mathcal{E}$ is 1.
   $\mathcal{E}$ contains by definition only a single object of size 0.
2. The EGF of $\mathcal{Z}$ is z.
   $\mathcal{Z}$ contains by definition only a single object of size 1.
3. The EGF of $\mathcal{A} + \mathcal{B}$ is $A(z) + B(z)$, given $\mathcal{A} \cap \mathcal{B} = \emptyset$.
   If $\mathcal{A} \cap \mathcal{B} = \emptyset$ then we have either an element from $\mathcal{A}$ or from $\mathcal{B}$.
   Count elements of size n in $\mathcal{A}$ then in $\mathcal{B}$ and add them up.

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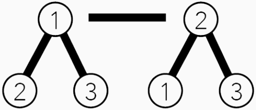
Let $\mathrm{c} \in \mathcal{A} \times \mathcal{B}$, size(c)= n.
Then c is a combination of a and b,
where $\mathrm{a} \in \mathcal{A}$, size(a)= k and $\mathrm{b} \in \mathcal{B}$, size(b)= $\mathrm{n} - \mathrm{k}$.

$\Rightarrow$ True for OGF's.

Observation

The EGF of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is $\mathrm{A(z)} \cdot \mathrm{B(z)}$.

When we combine objects, we have to relabel them...

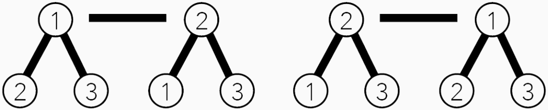


The Symbolic Method

Observation

The EGF of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is $\mathrm{A(z)} \cdot \mathrm{B(z)}$.

But we can't just relabel randomly, otherwise we doublecount!



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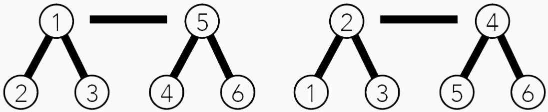
The EGF of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is $\mathrm{A}(\mathrm{z}) \cdot \mathrm{B}(\mathrm{z})$.

Relabeling has to be order-preserving

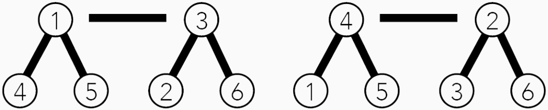


The Symbolic Method

Observation

The EGF of $\mathcal{C} = \mathcal{A} \times \mathcal{B}$ is $A(z) \cdot B(z)$.

Choose k labels for left object, use the other (n-k) labels for right object



$$C_n = \sum_{k=0}^{n} \binom{n}{k} A_k B_{n-k} = n! \sum_{k=0}^{n} \frac{A_k}{k!} \cdot \frac{B_{n-k}}{(n-k)!}$$

Sequence Construction

$$\mathcal{A} = \mathcal{Z}$$

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$$\mathcal{A} = \mathcal{Z} \times (\mathcal{E} + \mathcal{A})$$

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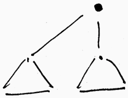
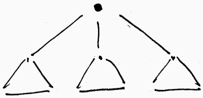


$$\mathcal{A} = \mathcal{Z} \times (\mathcal{E} + \mathcal{A} + \mathcal{A} \times \mathcal{A})$$

Sequence Construction



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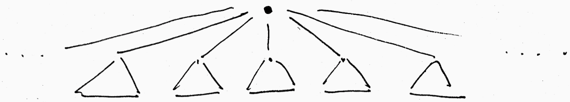
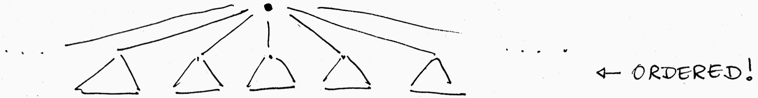


$$\mathcal{A} = \mathcal{Z} \times (\mathcal{E} + \mathcal{A} + \mathcal{A} \times \mathcal{A} + \mathcal{A} \times \mathcal{A} \times \mathcal{A} + \dots)$$

The Symbolic Method

Sequence Construction



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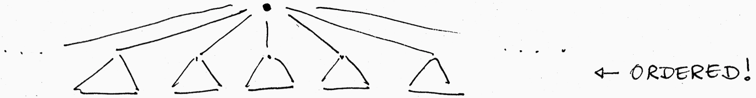

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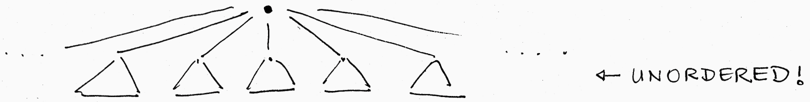
$$\mathrm{A(z)} \;=\; \mathrm{z} \cdot \sum_{\mathrm{n} \geq 0} \mathrm{A(z)^n} \;=\; \mathrm{z} \cdot \frac{1}{1 - \mathrm{A(z)}}$$

Set Construction



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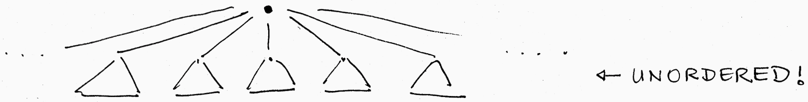
The Symbolic Method

Set Construction



$$\mathcal{A} = \mathcal{Z} \times \mathrm{SET}(\mathcal{A})$$

$$\mathrm{A(z)} = \mathrm{z} \cdot (1 + \mathrm{A(z)} + \frac{\mathrm{A(z)}^2}{2} + \frac{\mathrm{A(z)}^3}{3!} + \frac{\mathrm{A(z)}^4}{4!} \ldots) = \mathrm{z} \cdot \exp(\mathrm{A(z)})$$

Summary

If we can (literally) describe a combinatorial class in terms of "and, or, set, sequence", we can immediatly derive the generating function!

Construction	Symbolic	EGF
Sum	$\mathcal{A} + \mathcal{B}$	$\mathrm{A(z)} + \mathrm{B(z)}$
Product	$\mathcal{A} \times \mathcal{B}$	$\mathrm{A(z)} \cdot \mathrm{B(z)}$
Sequence	$\mathrm{SEQ}(\mathcal{A})$	$(1 - \mathrm{A(z)})^{-1}$
Set	$\mathrm{SET}(\mathcal{A})$	$\exp(\mathrm{A(z)})$

Analysis of the EGF

Singularity Analysis

Theorem (Cauchy-Hadamard)

Let $A(z) = \sum_{n \geq 1} A_n (z - c)^n$ be a power series with $A_n, c \in \mathbb{C}$. Then the radius of convergence of A at the point c is given by

$$\frac{1}{R} = \limsup_{n \to \infty} |A_n|^{\frac{1}{n}}$$

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$$|A_n| \sim R^{-n} \phi(n)$$

$\Longrightarrow$ "Singularity analysis"

Cauchy Integral Formula

Theorem (Cauchy Integral Formula)

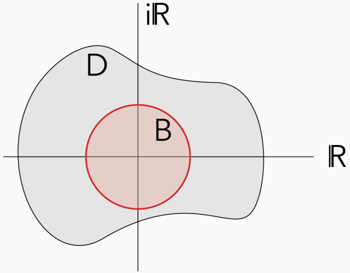
Let f be a holomorph function in an open subset $D \subseteq \mathbb{C}$ and let $\overline{B_r(c)} \subseteq D$. Then it holds for all $z \in B_r(c)$:

$$f(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(y)}{y - z} dy$$

As a consequence, f is analytic in $B_r(c)$ and for its Taylor series $f(z) = \sum f_n (z - c)^n$ it holds that

$$f_n = \frac{1}{2\pi i} \int_{\partial B} \frac{f(y)}{(y - c)^{n+1}} dy$$

Cauchy Integral Formula



If f is "nice" in D then it has Taylor series $f(z) = \sum f_n (z-c)^n$ in B and

$$f_n = \frac{1}{2\pi i} \int_{\partial B} \frac{f(y)}{(y-c)^{n+1}} dy$$

$\Longrightarrow$ Basis for most basic theorems in analytic combinatorics

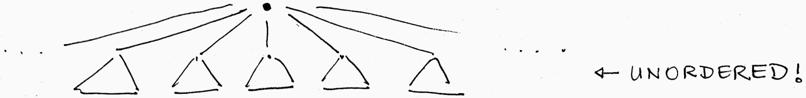
Lagrange Inversion Theorem

Theorem (Lagrange Inversion Theorem)

Let $\phi(u) = \sum_{k \geq 0} \phi_k u^k$ be a power series of $\mathbb{C}[[u]]$ with $\phi_0 \neq 0$. Then, the equation $A = z\phi(A)$ admits a unique solution in $\mathbb{C}[[u]]$ whose coefficients are given by

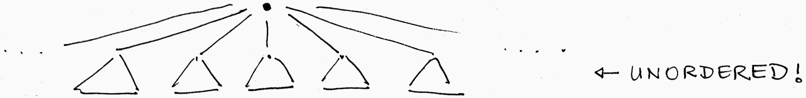
$$A(z) = \sum_{n \geq 1} A_n z^n, \quad \text{where } A_n = \frac{1}{n}[u^{n-1}]\phi(u)^n$$

Cayley's formula



$$\mathcal{A} = \mathcal{Z} \times \mathrm{SET}(\mathcal{A}) \quad \Longrightarrow \quad A(z) = \sum_{n \geq 0} \frac{A_n}{n!} z^n = z \cdot \exp(A(z))$$

Cayley's formula



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By Lagrange Inversion Theorem:

$$\begin{aligned} \frac{A_n}{n!} &= [z^n]A(z) = \frac{1}{n}[u^{n-1}] \exp(u)^n \\ &= \frac{1}{n}[u^{n-1}] \sum_{n \geq 0} \frac{(nu)^n}{n!} = \frac{n^{n-1}}{n(n-1)!} = \frac{n^{n-1}}{n!} \end{aligned}$$

Reference

Find out more:

Philippe Flajolet and Robert Sedgewick, Analytic Combinatorics, Cambridge University Press (2009).

Available online: http://algo.inria.fr/flajolet/Publications/book.pdf