Ultrafilters and Compactness of Topological Spaces

Diego Battistelli

April 2019

Why did they introduce filters and ultrafilters?

- They were useful to generalize the notion of convergence in a topological space.

$$\text{Classic convergence} \leftrightsquigarrow \text{Filter convergence}$$

Where are they used?

- Topology (we will see how compactness is linked to the convergence via a filter)
- Model theory (to build ultraproducts, that help us in the construction of non-standard models)
- Geometric group theory (to build asymptotic cones)
- **Bonus**: In Gödel's ontological proof of God's existence, the *positive properties* that a God should have, form an ultrafilter!

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Filters and Ultrafilters

Definition

Let I be a set and $F \subseteq \mathcal{P}(I)$.

$$\left.\begin{array}{l} 1.\ I \in F \\ 2.\ X \in F \wedge Y \in F \implies X \cap Y \in F \\ 3.\ X \in F \wedge X \subseteq Y \implies Y \in F \end{array}\right\} F \text{ is a filter}$$

We say that the filter F is an ultrafilter if we also have:

$$4.\ X \subseteq I \implies X \in F \vee I \setminus X \in F$$

(...and F becomes suddenly D, because that's the common letter for ultrafilters!)

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Example

Let I be a set and let

$$F := \{X \subseteq I : I \setminus X \text{ is a finite set}\}$$

F is called "the cofinite filter" (or "Fréchet filter") on the set I.

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Let I be a set, $i \in I$ and let

$$D(i) := \{X \subseteq I : i \in X\}$$

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Notation: If a filter contains $\emptyset$, then obviously the filter is $\mathcal{P}(I)$ and it is called *improper filter*. If the filter doesn't contain $\emptyset$, then it is said to be *proper*.

Some facts:

- Every proper filter on I contains the cofinite filter on I.
- Every proper filter is contained in an ultrafilter.
- Let D be an ultrafilter on a set I and $X_1, \ldots, X_n \subseteq I$.
  If $X_1 \cup \ldots \cup X_n$ is a big set for D, then one of the X_i is big for D.

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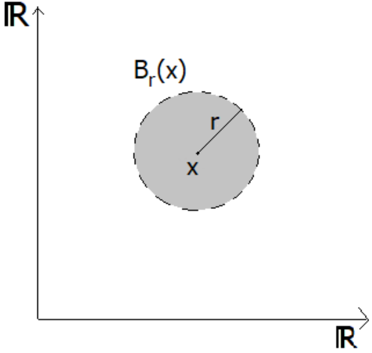
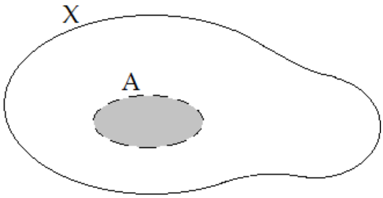
What are open sets?

In $\mathbf{R}^2$



In an abstract space X



Just some topology...

Definition

If X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$, then $\mathcal{T}$ is a topology on X if

- $\emptyset$, $X \in \mathcal{T}$;
- $A_1, \ldots, A_n \in \mathcal{T} \implies \bigcap_{i=1}^n A_i \in \mathcal{T}$;
- $\{A_i\}_{i \in I} \subseteq \mathcal{T} \implies \bigcup_{i \in I} A_i \in \mathcal{T}$.

Some notations and definitions:

- The element of $\mathcal{T}$ are called *open sets*.
- An *open cover of* X is a family $\{U_i\}_{i \in I}$ of open sets such that $X \subseteq \bigcup_{i \in I} U_i$.
- A space X is *compact* if every open cover of X has a finite subcover (e.g. the set $[0, 1] \subseteq \mathbb{R}$ is compact).

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Example

Let's take $C[0,1] := \{f : [0,1] \longrightarrow \mathbb{R} \mid f$ is a continuous function$\}$.

- $\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|$
- $d(f,g) := \|f - g\|_\infty$
- $B_r(f) := \{g \in C[0,1] \mid d(f,g) < r\}$

Remark: spaces of functions are really interesting when you want to study compactness!

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Products

What is the cartesian product?

- The product of two sets X and Y is just the set of the ordered pairs (x, y) such that $x \in X$ and $y \in Y$, which we denote with $X \times Y$.
- The cartesian product of an arbitrary family of sets $\{X_j\}_{j \in J}$ is the set $\prod_{j \in J} X_j$ containing all the sequences $(x_j)_{j \in J}$ such that $x_j \in X_j$.

Can we give a topology to this product?

- Obviously, otherwise this talk had to finish here! I'm sorry!
- We can consider the following open sets

$$\prod_{j \in J} A_j \quad \text{s.t.} \quad \begin{array}{r} A_j \text{ is an open set of } X_j \ \forall j \in J \\ \text{and } A_j = X_j \text{ for almost every } j \in J. \end{array}$$

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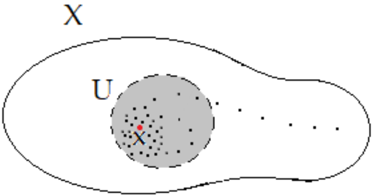
D-convergence

By *sequence of elements of* X we mean $(x_i)_{i\in I}$ s.t. $x_i \in X$

- If $I = \mathbb{N}$, that's a *classical* sequence.

Definition

Let X be a TS, D be a filter on a set I, $(x_i)_{i\in I} \subseteq X$, $x \in X$. We say that $(x_i)_{i\in I}$ D-converges to x if there is a *big* number of elements of the sequence *around* x (i.e. for every U open set containing x, the set $\{i \in I : x_i \in U\} \in D$).



D-convergence

- If $I = \mathbb{N}$ and D is the cofinite filter, then a sequence D-converges to some $x \in X$ iff it converges to x in the classical way.

Definition

Let X be a TS, D an ultrafilter on I. X is D-compact if every sequence $(x_i)_{i \in I} \in X$ is D-convergent to some element of X.

We will use this notion to characterize the classical compactness. Using this characterization, we will prove the following theorem:

Theorem (Tychonoff)

The product of compact spaces is still compact.

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D-compactness

We will use the following two theorems:

Theorem (1)

Let X be a TS.
X is compact iff X is D-compact for every ultrafilter D.

Theorem (2)

Let D be an ultrafilter.
The product of D-compact spaces is still D-compact.

From this two results we have Tychonoff's theorem as a simple corollary!

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Sketch of the proof.

- By contradiction, there is a sequence $(x_i)_{i \in I}$ which doesn't D-converge in X (i.e. for every element x of X we find an open set U_x containing x such that there is a *small* number of elements of the sequence in it).
- Consider the open cover $\{U_x\}_{x \in X}$ and extract the finite subcover $U_{x_1}, \dots, U_{x_n}$ since X is compact.

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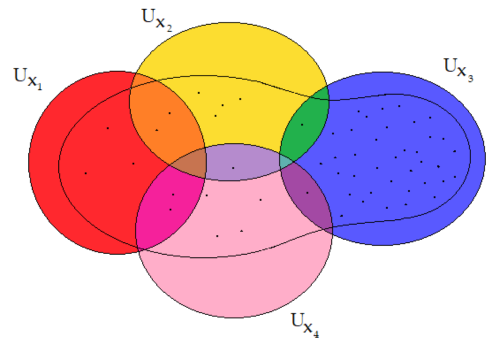
Sketch of the proof.

All the elements of the sequence lie in the union of these n sets, because they cover X. Thus, the indices of the elements which are in this union form a *big* set (it's exactly I).



Hence, one of these sets has to be a *big* set! Contradiction! □

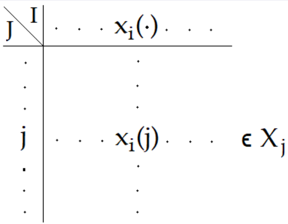
Theorem (2)

Theorem (2)

Let D be an ultrafilter on a set I.
The product of D-compact spaces is still D-compact.

Sketch of the proof.

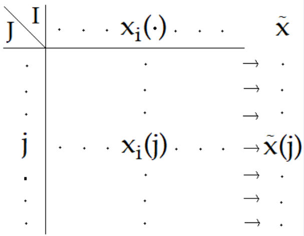
- Let $\{X_j\}_{j \in J}$ be a family of D-compact spaces. A sequence in the product $\prod_{j \in J} X_j$ is $(x_i)_{i \in I}$ s.t. every x_i is a sequence $(x_i(j))_{j \in J}$.



- Fix $j \in J$. Thus, $(x_i(j))_{i \in I}$ D-converges in X_j to some $\tilde{x}(j)$ since every X_j is D-compact.

Sketch of the proof.

- Consider the sequence $\tilde{x} = (\tilde{x}(j))_{j \in J}$, i.e.



Let's show that $(x_i)_{i \in I}$ D-converges to $\tilde{x}$.

- Open set containing $\tilde{x}$ in $\prod_{j \in J} X_j$ is $\prod_{j \in J} A_j$ (A_j are open sets and almost all are X_j).
- $\tilde{x} \in \prod_{j \in J} A_j \implies \tilde{x}(j) \in A_j$. So A_j is an open set containing this element in X_j, which is D-compact! Hence there is a big number of elements of $(x_i(j))_{i \in I}$ in A_j for every j.

Theorem (2)

Sketch of the proof.

- For almost every $j \in J$ we have $A_j = X_j$, hence this set of indices is exactly I (which is big).
- For the other $j \in J$, which are in finite number, the set of indices is *big*, so the intersection is still *big*.
- Then, the intersection of all the indices is *big*, i.e.

$$\bigcap_{j \in J} \{i \in I : x_i(j) \in A_j\} = \{i \in I : x_i(j) \in A_j \ \forall j \in J\}$$

$$= \{i \in I : x_i \in \prod_{j \in J} A_j\} \text{ is } \textit{big}$$

Which means that a big number of elements of our initial sequence is around $\tilde{x}$.

$\square$

Back to Tychonoff

Proof (Tychonoff).

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$\Downarrow$ (1) (1) $\Uparrow$

$\{X_j\}_{j\in J}$ are D-compact spaces for every UF D $\overset{(2)}{\Longrightarrow}$ $\prod_{j\in J} X_j$ is a D-compact space for every UF D

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Some applications of Tychonoff's theorem:

- In fuctional analysis: to prove many results such as Banach-Alaoglu's theorem or Ascoli-Arzela's theorem.
- In logic: to prove the compactness of the first-order logic.
- **Bonus**: Tychonoff's theorem is equivalent to the Axiom of Choice!

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Thanks for your attention!

Grazie!

Děkuji!

Merci!

Danke!

Köszönöm!

Hvala!

...and so on...