No-three-in-line-problem on a torus

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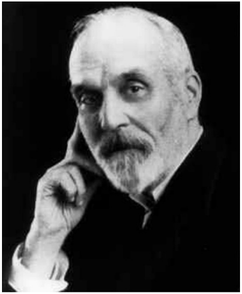
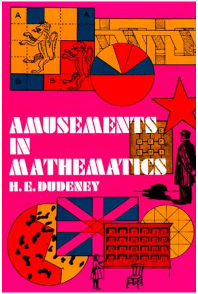




Puzzle 317

Place two pawns in the middle of the chessboard, one at Q4 and the other at K5. Now, place the remaining fourteen pawns (sixteen in all) so that no three shall be in a straight line in any possible direction.

History - No-three-in-line-problem

No-three-in-line-problem

How many points can be placed on an $n \times n$ grid so that no three points are collinear.

- Still unsolved for general n.

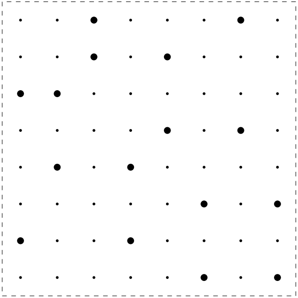


Figure: Dudeney's solution for the chessboard (8×8 grid).

Discrete torus $T_{m \times n}$

Cartesian product $\{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \subset \mathbb{Z}^2$.

Line on $T_{m \times n}$ is an image of a line in $\mathbb{Z}^2$ under a mapping which maps a point $(x, y) \in \mathbb{Z}^2$ to the point $(x \mod m, y \mod n)$.

Line in $\mathbb{Z}^2$ $\{(b_1, b_2) + k(v_1, v_2); k \in \mathbb{Z}\}$, where $\gcd(v_1, v_2) = 1$.

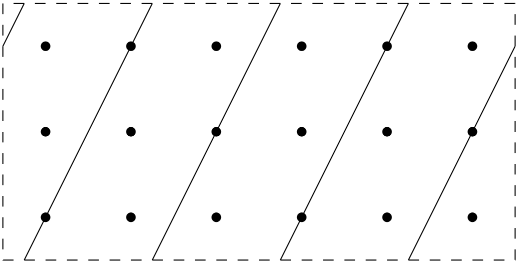
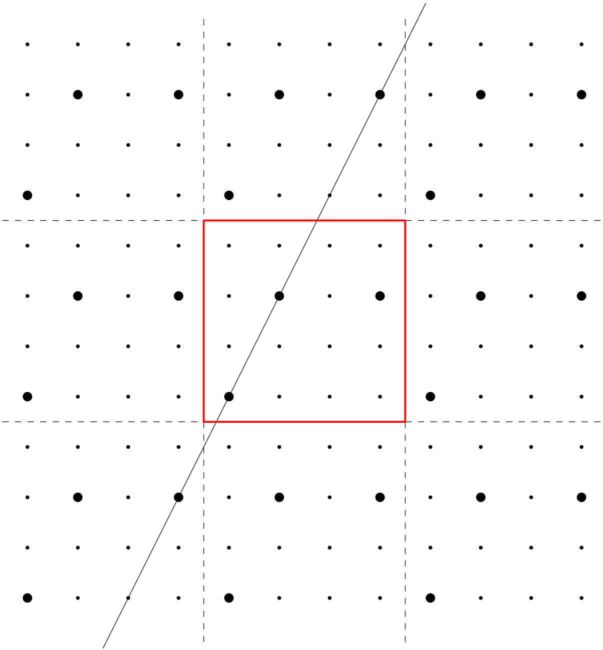
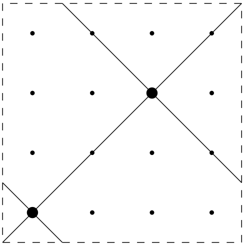


Figure: $T_{3 \times 6}$

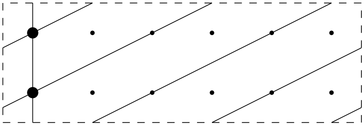


Discrete torus $T_{m \times n}$

- More lines between two points.



- A line is a proper subset of another line.



No-three-in-line-problem on a torus

No-three-in-line-problem on a torus [Fowler at al. 2012]

How many points can be placed on a discrete torus $T_{m\times n}$ of size $m \times n$ so that no three points are collinear.

- Let $\tau_{m,n}$ denote such maximum number of points.

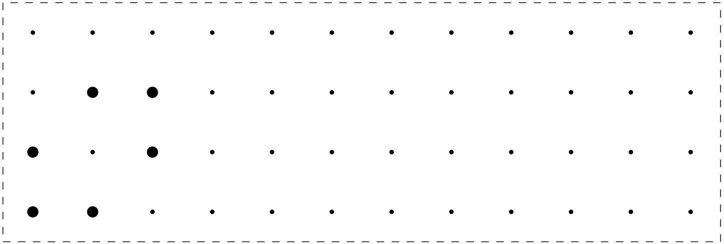


Figure: $\tau_{4,12} = 6$.

Algebraic viewpoint

$T_{m \times n}$ is an abelian group $\mathbb{Z}_m \times \mathbb{Z}_n$.

Line on $T_{m \times n}$ is a coset of a cyclic subgroup.

Question

What is $\tau_{m,n}$ for relative primes m, n?

- $\tau_{m,n} = 2$ by the Chinese remainder theorem.

General upper bound

Theorem

$\tau_{m,n} \leq 2\gcd(m,n)$.

Proof.

- The length of the line ℓ generated by $(1,1)$ is $|\ell| = \mathrm{lcm}(m,n)$.
- Then by Lagrange's theorem $[T_{m\times n} \colon \ell] = \frac{|T_{m\times n}|}{|\ell|}$.
- Therefore we have $[T_{m\times n} \colon \ell] = \frac{mn}{\mathrm{lcm}(m,n)} = \gcd(m,n)$ disjoint lines covering the whole torus. Each of these lines may contain at most two points.

$\square$

Basic relations

Theorem

$\tau_{m,n} \leq \tau_{xm,yn}$.

Sketch of proof.

- The image of each line from $T_{xm \times yn}$ is a line on $T_{m \times n}$.
- Therefore each set of points from $T_{m \times n}$ with no three of them collinear can be used on the bigger torus $T_{xm \times yn}$ so that no three points are collinear.

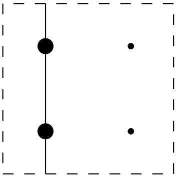
Basic relations

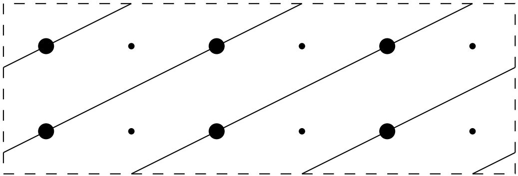
Theorem

If $\gcd(x, y) = \gcd(m, y) = \gcd(n, x) = 1$ *then* $\tau_{m,n} = \tau_{xm,yn}$.

Sketch of proof.

- The preimage of each line from $T_{xm \times yn}$ is a line on $T_{m \times n}$.
- Therefore the image of each set of points from $T_{xm \times yn}$ with no three of them collinear can be used on the smaller torus $T_{m \times n}$ so that no three points are collinear.





Other known results
For powers of primes

- $\tau_{p,p} = p + 1$
- $\tau_{p^a,p^{(a-1)p+2}} = 2p^a$
- $\tau_{2^a,2^{2a-1}} = 2^{a+1}$

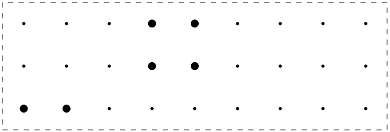


Figure: Example of the construction for $T_{3\times 9}$.

Therefore the problem is solved for m, n such that $\gcd(m, n) = p$ for a prime p.

Upper bound

- $\tau_{p^a,p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$

Sequences

If we fix one coordinate of a torus, we get the sequence $\tau_{z,1}, \tau_{z,2}, \tau_{z,3}, \ldots$ for $z \geq 2$, which we denote σ_z.

z	1	2	3	4	5	6	7	8	9	10	11	12	13	…
2	2	4	2	4	2	4	2	4	2	4	2	4	2	…
3	2	2	4	2	2	4	2	2	6	2	2	4	2	…
4	2	4	2	6	2	4	2	8	2	4	2	6	2	…
5	2	2	2	2	6	2	2	2	2	6	2	2	2	…
6	2	4	4	4	2	8	2	4	6	4	2	8	2	…

Table: Initial values of $\tau_{z,n}$.

- The potential maximum of the sequence is $2z$. Since $\tau_{m,n} \leq 2\gcd(m, n)$.

Sequences

Question

Does the sequence σ_z reach its potential maximum $2z$?

- It is true for σ_p for a prime p.
  Since $\tau_{p^a, p^{(a-1)p+2}} = 2p^a$.
- Open for general z.

Question

Is the sequence σ_z periodic?

- Yes! For each z.
- However, the proof is purely existence.

Periodicity

Theorem

Let p be a prime and let us denote $m := \min\{x; \sigma_{p^a}(x) = 2p^a\}$. The sequence σ_{p^a} is periodic with the period m.

Strategy.

We know:

- $\sigma_z(n) \leq \sigma_z(xn)$.
- $\sigma_z(xn)$ if $\gcd(x, z) = 1$.

Example for σ_4.

- We want to show $\tau_{4,12} = \tau_{4,28}$.
- It is true since $\tau_{4,12} = \tau_{4,4}$ and also $\tau_{4,28} = \tau_{4,4}$.

Periodicity

Theorem

Let p be a prime and let us denote $m := \min\{x; \sigma_{p^a}(x) = 2p^a\}$. The sequence σ_{p^a} is periodic with the period m.

Proof.

We consider two cases:

- $x < p^b$.
  In this case $x + \alpha p^b = p^l(r + \alpha p^{b-l})$. Since $b - l \geq 1$, we get $\gcd(r + \alpha p^{b-l}, p) = 1$. Therefore $\sigma_{p^a}(p^l(r + \alpha p^{b-l})) = \sigma_{p^a}(p^l) = \sigma_{p^a}(x)$.
- $x = p^b$.
  In this case $x + \alpha p^b = p^b(1 + \alpha)$. Since $\sigma_{p^a}(p^b) = 2p^a = \max \sigma_{p^a}$, we know $\sigma_{p^a}(p^b) = \sigma_{p^a}(hp^b)$ for any $h > 0$ and hence also for $h = (1 + \alpha)$.

Therefore $\sigma_{p^a}(x) = \sigma_{p^a}(x + \alpha p^b)$ for any $x \in \{1, \ldots, p^b\}$. $\square$

Periodicity

"Conjecture"

The least period for σ_6 is 108.

- Is it true that $\sigma_6(2 \cdot 3^j) = 6$ for all $j \geq 2$?

$\gcd(n, 2^2 \cdot 3^3)$	2^0	2^1	2^2
3^0	2	4	4
3^1	4	8	8
3^2	6	10	12
3^3	6	12	12

Table: Values of $\sigma_6(n)$ according to $\gcd(n, 2^2 \cdot 3^3) = 2^i \cdot 3^j$.

Thank you!

Summary:

- $\tau_{m,n} = 2$ if $\gcd(m,n) = 1$.
- $\tau_{m,n} \leq 2\gcd(m,n)$.
- $\tau_{m,n} \leq \tau_{xm,yn}$.
- $\tau_{m,n} = \tau_{xm,yn}$ if $\gcd(x,y) = \gcd(m,y) = \gcd(n,x) = 1$.

- $\tau_{p,p} = p + 1$.
- $\tau_{p^a,p^{(a-1)p+2}} = 2p^a$.
- $\tau_{2^a,2^{2a-1}} = 2^{a+1}$.
- $\tau_{p^a,p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$.

- The sequence $\tau_{z,1}, \tau_{z,2}, \tau_{z,3}, \dots$ is periodic for all z.

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