5^{th} Day. May 25, 2011 (3+2=5 hours) [Some Statistical Data Analysis]

: (i) Decomposition of Interest Rate Time series data.(Smoothing).
And JPyen/USdollar currency rates.

: (ii) Hedge Fund Returns.. VaR for HF returns
: (iii) Replication of Hedge Funds

: (iv) Trials for Data Analysis based on Brownian Nonparametric Statistics.

An additional topic.
: (v).Spacing estimation of a density function.

References

R.Miura (11). "Decomposition of Japanese Yen Interest Rate Data Through Local Regression.“ (with R. Shibata),

Financial Engineering and the Japanese Markets, Vol.4, No.2, pp. 125-146, 1997

R.Miura (12). "On Financial Time Series Decompositions with Applications to Volatility" (with Kjell Doksum and Hiroaki Ymauchi)

Hitotsubashi Journal of Commerce and Management, Vol.35, No.1, pp 19-47, October 2000

R.Miura(13).“A Note on Statistical Models for Individual Hedge Fund Returns.”

(with Aoki and Yokouchi)

Mathematical Methods of Operations Research. Vol.69, Issue 3 (2009). Page 553. Springer.

It is popular to use a kernel method for estimating a probability density, but one can also use the differences of order statistics(spacings) to estimate a probability density.

R.Miura (14). "Spacing Estimation of the Asymptotic Variance of Trimmed Rank Estimators of Location"
Scandinavian Journal of Statistics, No.8, pp.48-54, 1981

R.Miura (15). "Spacing Estimation of the Asymptotic Variance of Rank Estimators"
Proceedings of Golden Jubilee Conference of Indian Statistical Institute: Statistics; Applications and New Directions, pp.391-404, December 1981

End-of-class

No homework today.

But, we take Photoes of us at the end of afternoon class.

Section 1. Swap Interest rates in Japan 1987--1993

: Decomposition of time series data.

: Locally weighted regression

: Use this twice.

: 1 year span.

: 1 month span.

2. Data

Data we are analyzing are 3 month, 6 month and 1 year Euro Yen interest rate series, and 3 year, 5 year, 7 year, and 10 year LIBOR swap rate series. All series are daily for the period; from the 1st of December, 1986 to the 16th of September, 1992. This data is now regarded as historic in a sense that the period of over heated Japanese economy, or the 'bubble' economy, is included. The length of each series is 2115, so that this study required a powerful computing environment since computer intensive smoothing techniques like *sabl* or *lowess* are used.

3. Decomposition

3.1. SABL DECOMPOSITION

As a preliminary analysis, we tried the *sabl* (Seasonal Adjustment at Bell Laboratories) decomposition procedure which is proposed by Cleveland and Devlin (1988) and implemented as a function `sabl` in S (Becker *et al.*, 1988). This *sabl* procedure is widely used as well as X-11 for time series analysis of real data. For example, Shiba and Takeji (1994), aiming at better prediction of asset price, applied this procedure to various price indices: TOPIX (Tokyo Stock Exchange Price Index), two specific stock prices, Yen/Dollar exchange rate, and long term bond price index. By *sabl*, the given time series is decomposed into three parts,

$$\text{original series} = \text{trend} + \text{seasonal} + \text{irregular}.$$

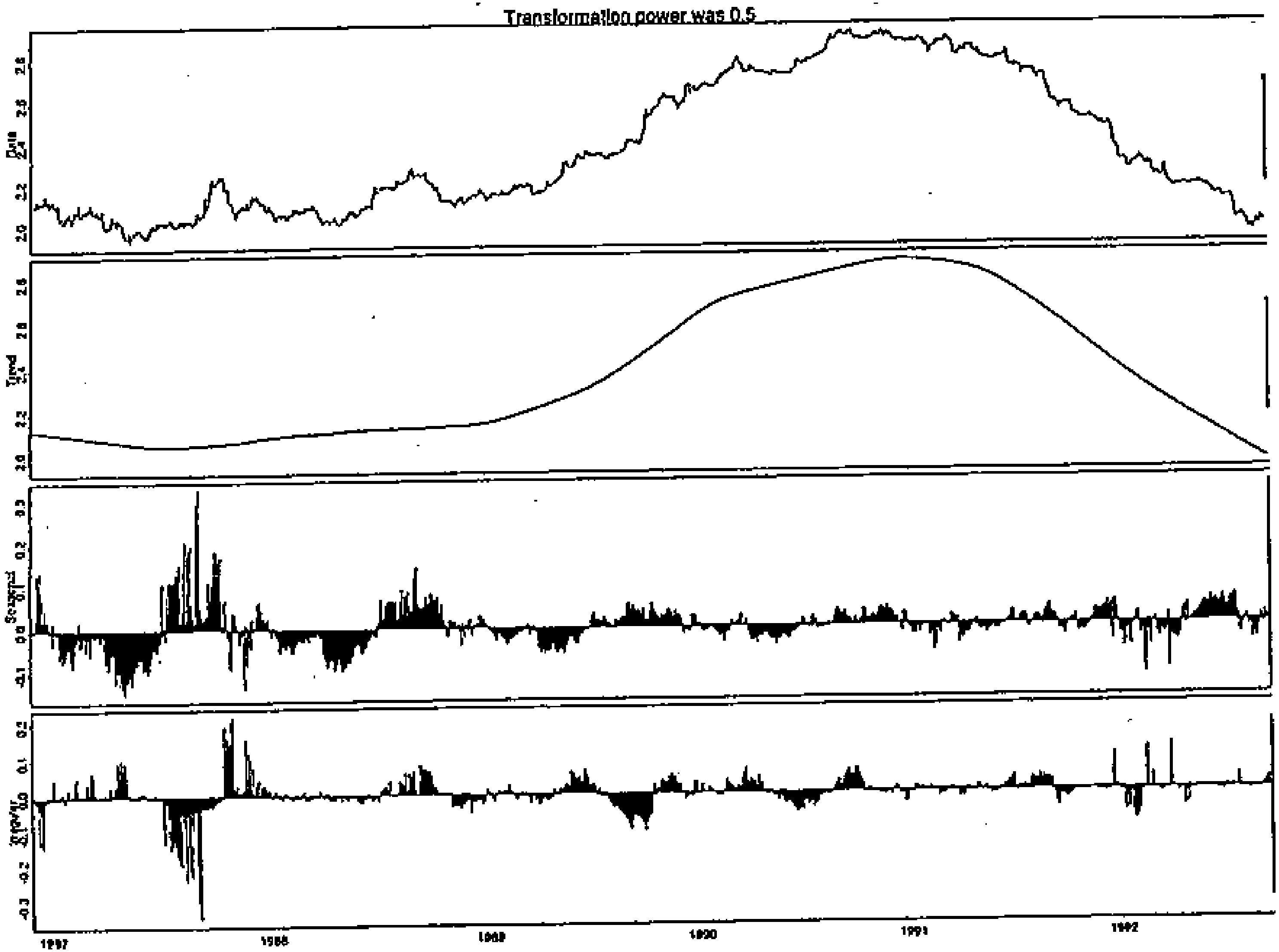


Figure 1. *sabl* decomposition of the 3 month Euro Yen interest rate series. The seasonal component on the third panel was obtained by smoothing over the same days of 3 neighboring years.

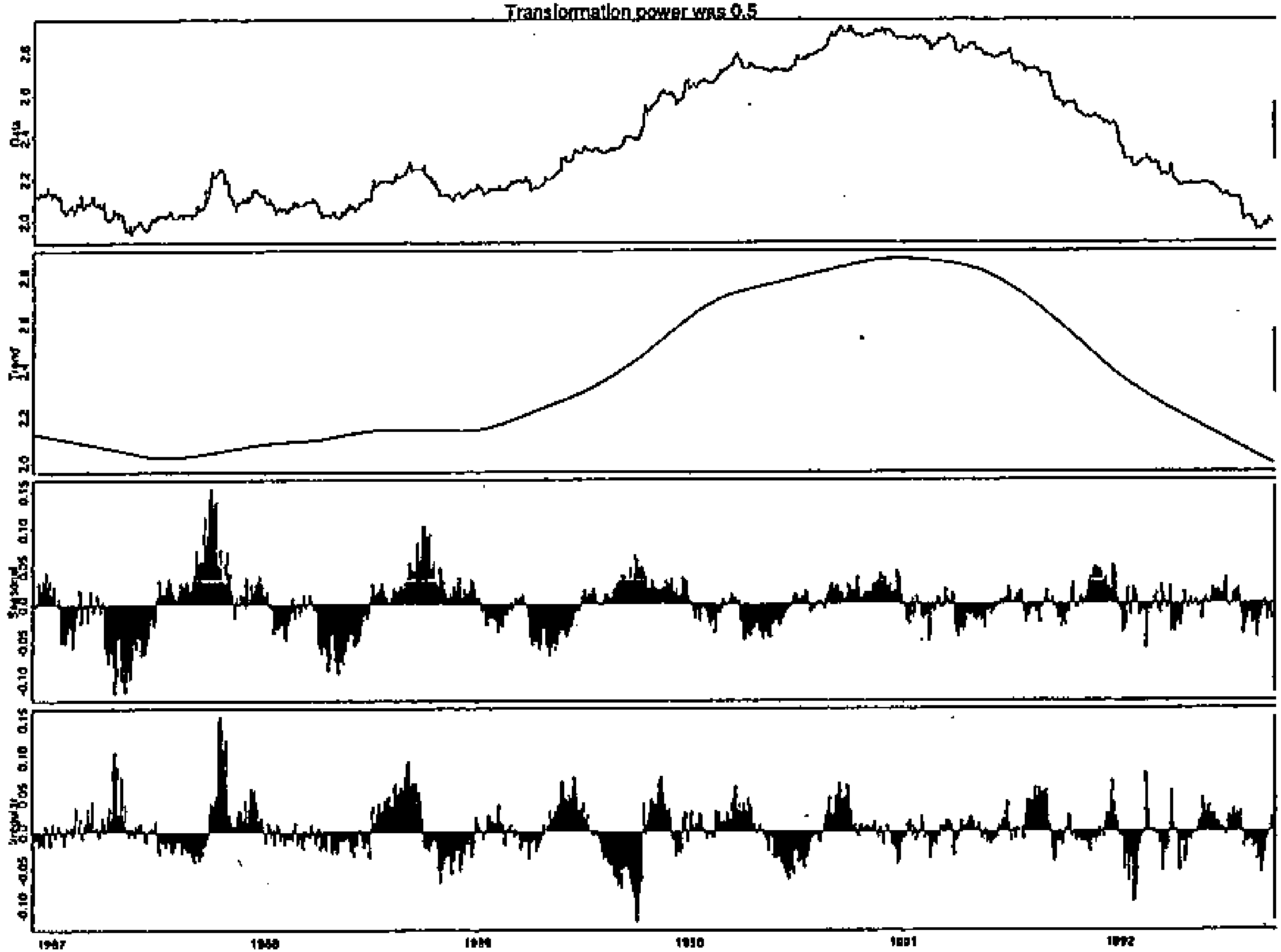


Figure 2. *sabl* decomposition of the 3 month Euro Yen interest rate series. The seasonal component on the third panel was obtained by smoothing over the same days of 7 neighboring years.

Our decomposition

Two step smoothing by Locally weighted regression

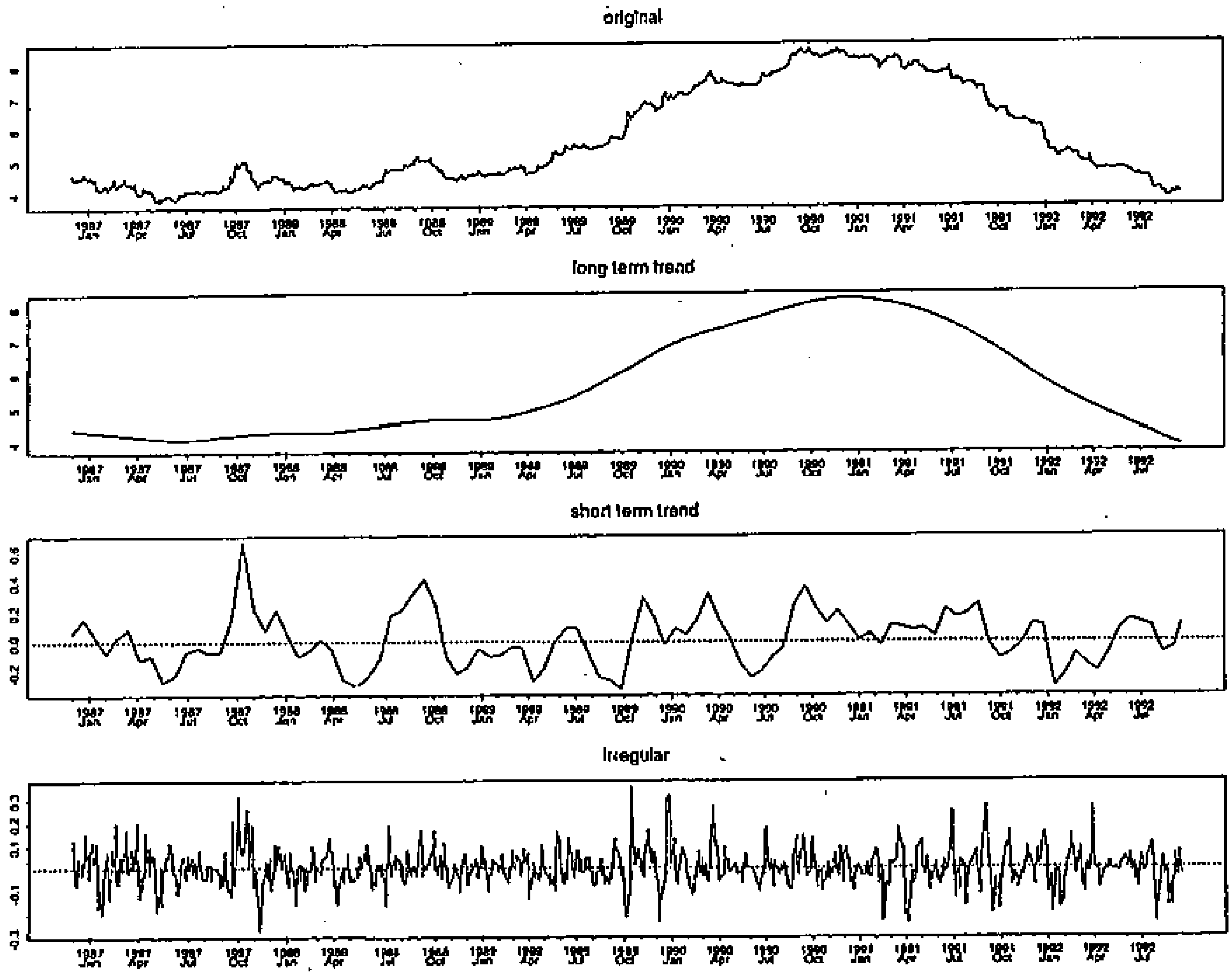


Figure 3. Decomposition of the 3 month Euro Yen interest rate series, shown on the top panel, into the following three components, the long term trend, the short term trend and the irregular series.

This Looks better than sabl for the data of our concern.

Let us look into the details of this.

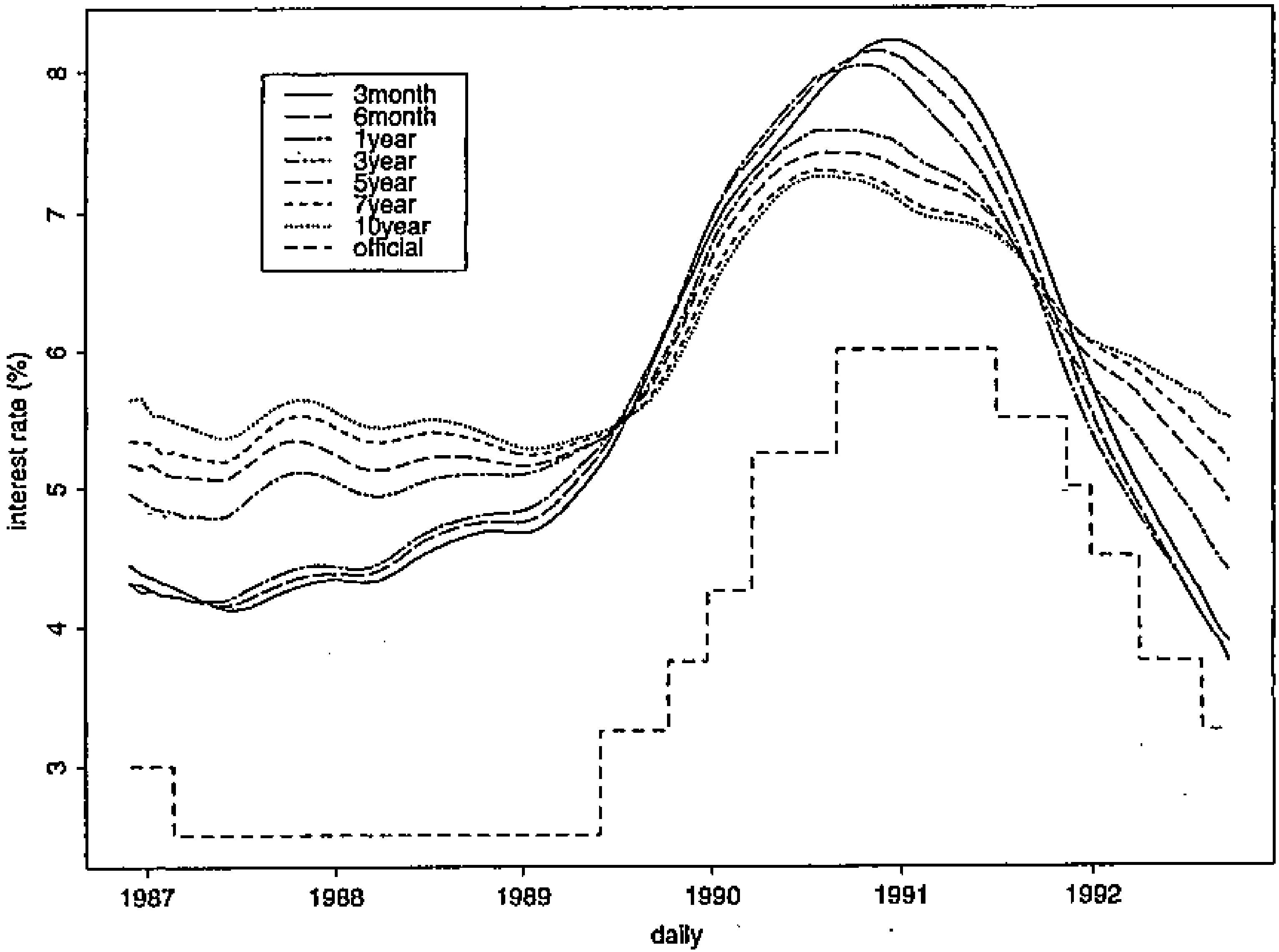


Figure 4. Long term trends and the official discount rate series.

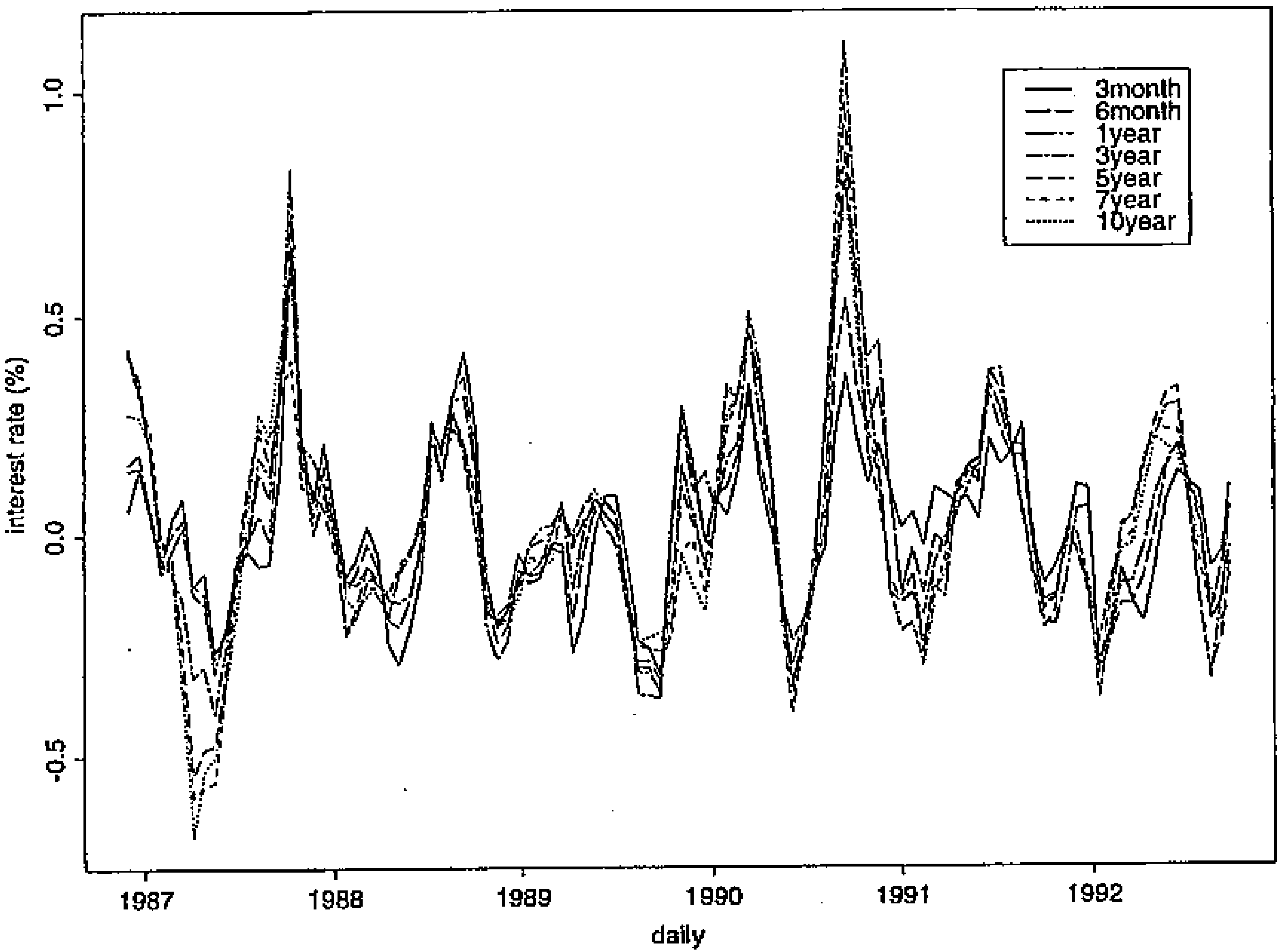


Figure 6. Short term trends

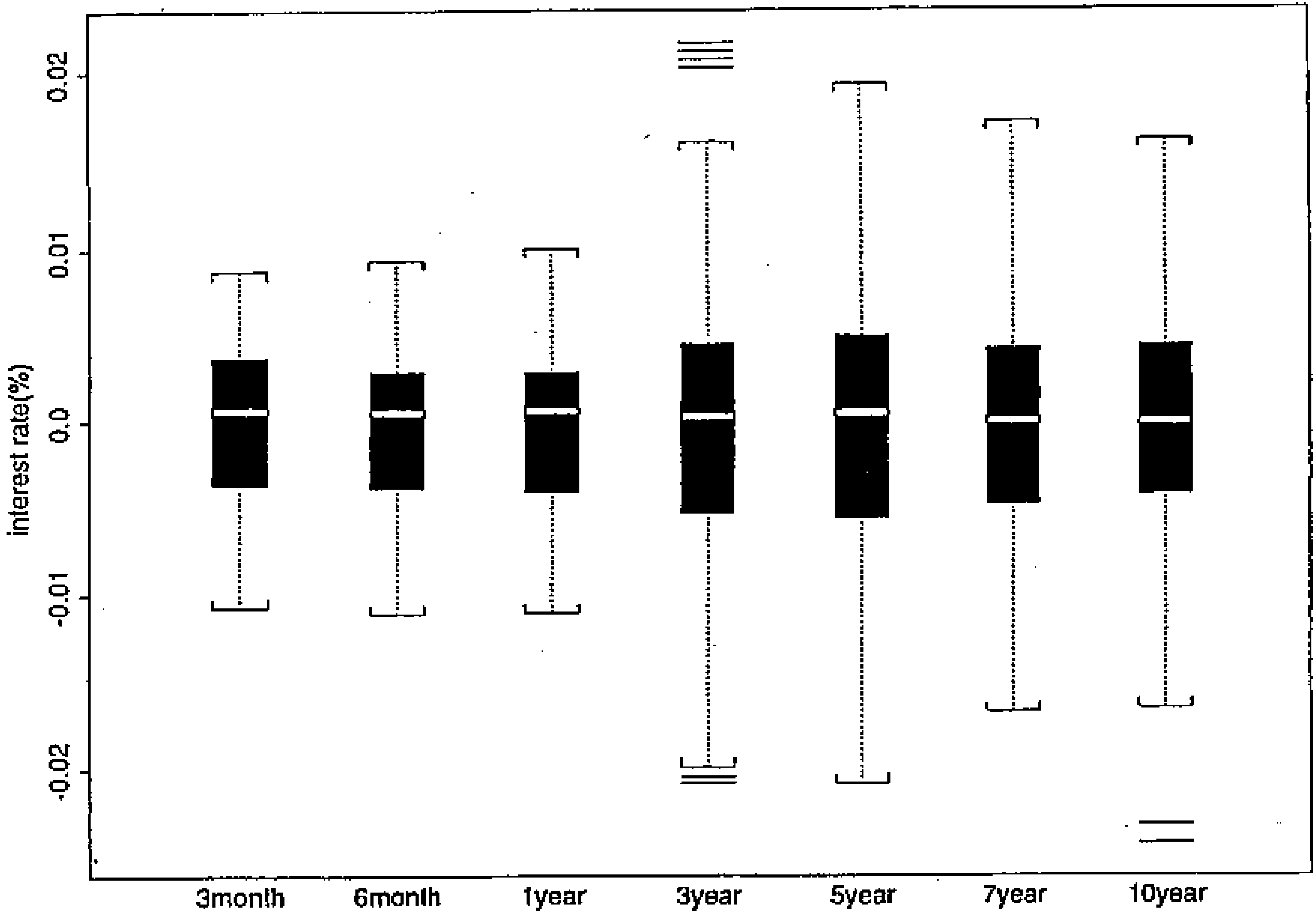


Figure 5. Boxplots of lag 1 differences of the long term trends.

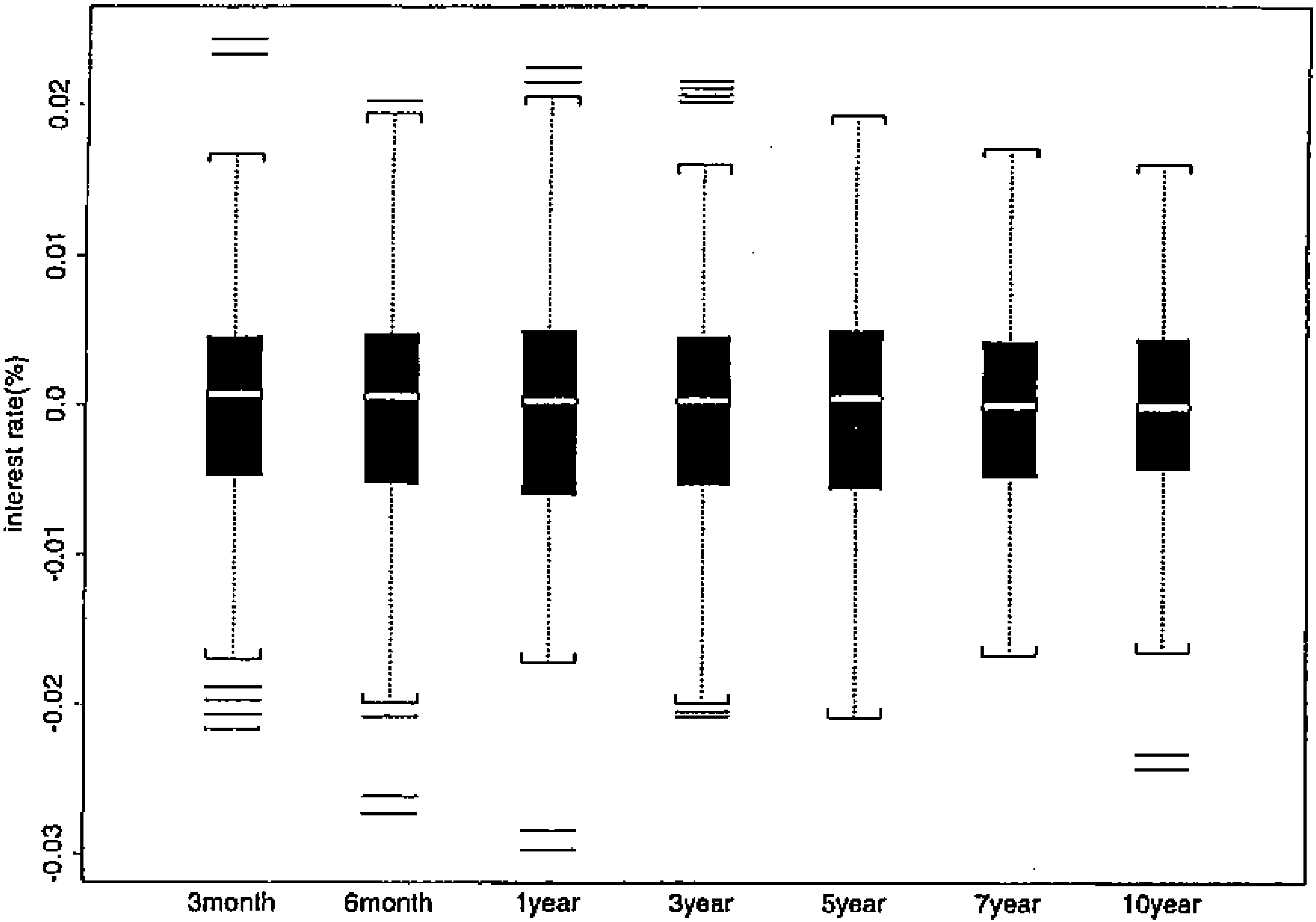


Figure 7. Boxplots of lag 1 differences of short term trends

$$Z_i(t) = L_i(t) + S_i(t) + I_i(t),$$

4.3. IRREGULAR

The seven irregular series obtained were combined into a multivariate time series $I(t) = (I_1(t), \ldots, I_7(t))^T$ and a multivariate autoregressive model MAR(2),

$$I(t) = AI(t-1) + BI(t-2) + \varepsilon(t),$$

is fitted to them. The order of the autoregression is selected so as to minimize AIC. Autocorrelations of the residuals $\varepsilon_i(t), i = 1, \ldots, 7$ are shown in Figure 9. There is no doubt on the orthogonality of each residuals as a time series, since all autocorrelations are between -0.06 and 0.05. This shows goodness of fit of the MAR(2) model.

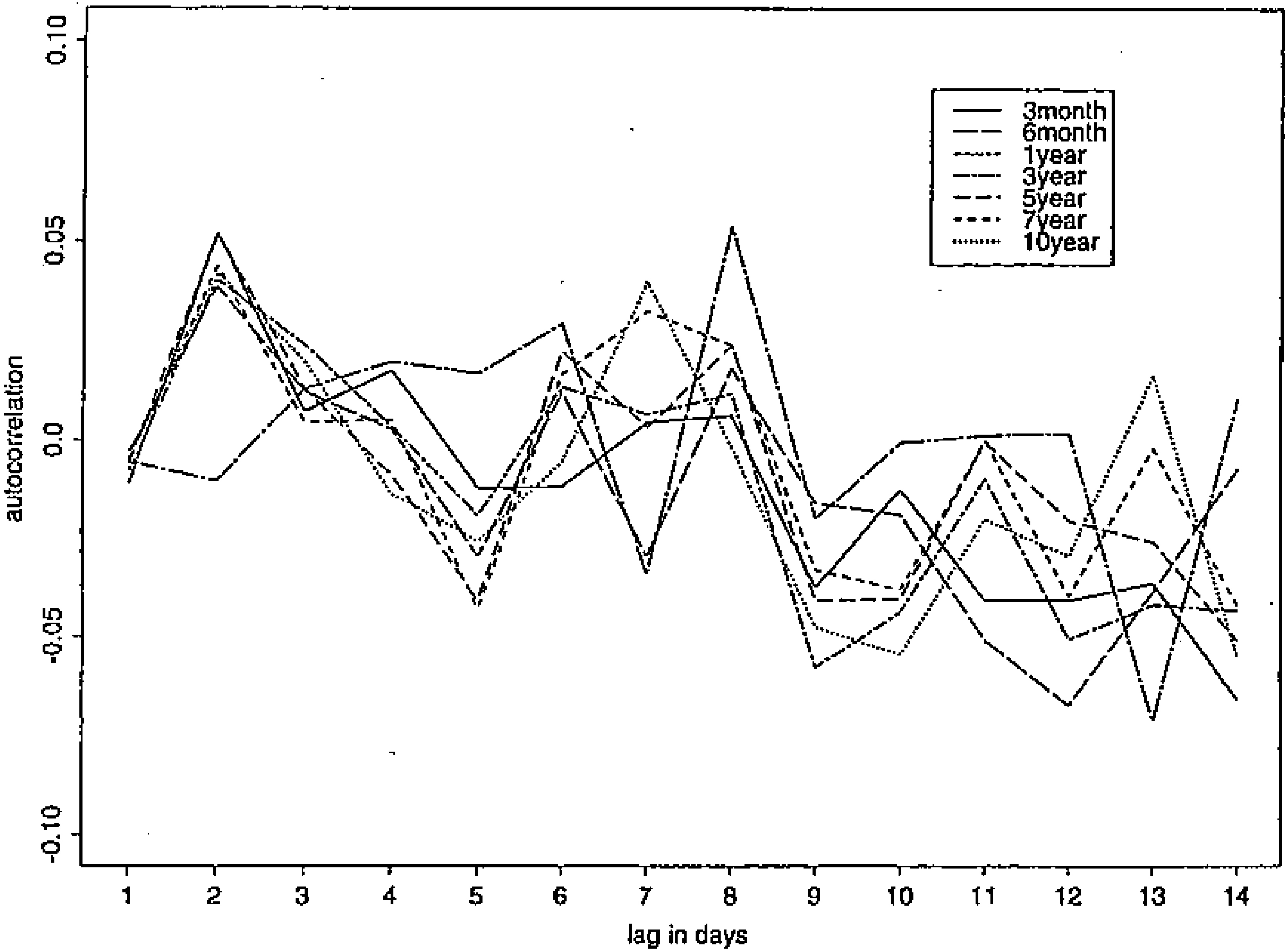


Figure 9. Autocorrelations of residuals

It is interesting to note that the coefficient matrix A in Table III shows that the 6 month rate significantly affects the 3 month rate and also affects other rates except the 10 years. This behavior is understandable since trading volume of 6 month interest rate is the biggest in the market. We could not find any good reason why all elements of the 6th column of the coefficient matrix B in Table IV are positive and significantly high except the first. In particular, the 2nd and the 3rd elements are significant compared with other elements of the matrix. This implies that the two days before rate of the 7 years affects the 6 month or the 1 year rate positively. It is said that bond future and the 7 year swap are very much related since bonds with 7 years maturity are usually the cheapest among the nominates of the cheapest-to-deliver (bonds with 7 to 10 years maturity) for the bond future trading in the market. Note that the portfolio of long term swaps are usually hedged with bond futures. It is also said that the issuing banks of 5 years maturity bonds are limited to the three banks, the Nippon Credit Bank, the Industrial Bank of Japan, and the Long Term Credit Bank of Japan so that the 5 year swap rate is affected and tends

Table III. Coefficient matrix A

3 months	6 months	1 year	3 years	5 years	7 years	10 years
0.693	0.381	0.019	−0.025	0.150	−0.062	−0.0936
0.141	0.769	0.016	0.115	0.031	−0.128	0.0026
0.053	0.245	0.603	0.071	0.086	−0.163	0.0515
0.021	0.201	−0.036	0.884	0.089	−0.137	0.0689
0.025	0.160	−0.037	0.134	0.790	−0.081	0.0946
0.000	0.133	−0.010	0.066	0.038	0.741	0.1001
0.036	0.086	−0.000	0.147	0.000	−0.086	0.8681

Table IV. Coefficient matrix B

3 months	6 months	1 year	3 years	5 years	7 years	10 years
−0.045	−0.0980	−0.04783	−0.0014	−0.12574	0.08	0.0454
−0.076	−0.0035	−0.01673	−0.1413	−0.04465	0.21	−0.0043
−0.074	−0.1159	0.12864	−0.0638	−0.09166	0.24	−0.0775
−0.091	−0.0472	0.01298	−0.0684	−0.06411	0.16	−0.1213
−0.073	−0.0554	0.03210	−0.0819	−0.00041	0.12	−0.1536
−0.045	−0.0595	0.00501	−0.0520	−0.00717	0.12	−0.1645
−0.058	−0.0449	0.00049	−0.1181	0.02371	0.12	−0.1237

Table V. Residual standard error

3 months	6 months	1 year	3 years	5 years	7 years	10 years
0.035	0.04	0.04	0.031	0.031	0.029	0.028

Table VI. Residual correlation matrix

	3 months	6 months	1 year	3 years	5 years	7 years	10 years
3 months	1.00	0.47	0.37	0.23	0.22	0.20	0.18
6 months	0.47	1.00	0.73	0.42	0.41	0.37	0.33
1 year	0.37	0.73	1.00	0.46	0.45	0.41	0.36
3 years	0.23	0.42	0.46	1.00	0.86	0.79	0.74
5 years	0.22	0.41	0.45	0.86	1.00	0.83	0.79
7 years	0.20	0.37	0.41	0.79	0.83	1.00	0.86
10 years	0.18	0.33	0.36	0.74	0.79	0.86	1.00

Table VII. Eigen values of residual covariance matrix

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
0.0044654	0.0016757	0.0008558	0.0004066	0.0002422	0.00013127	0.00010659

Table VIII. The first four eigen vectors of the covariance matrix of residuals

	u_1	u_2	u_3	u_4
3 months	0.2599885	−0.4014906	0.85685183	0.192176906
6 months	0.4779732	−0.4539444	−0.19496352	−0.726141302
1 year	0.4773444	−0.3520577	−0.45793555	0.658369742
3 years	0.3734923	0.3407595	0.05364768	0.008425985
5 years	0.3765874	0.3663068	0.05499962	−0.003553742
7 years	0.3264048	0.3614640	0.07069049	−0.018223919
10 years	0.2970443	0.3575489	0.08478813	−0.043822272

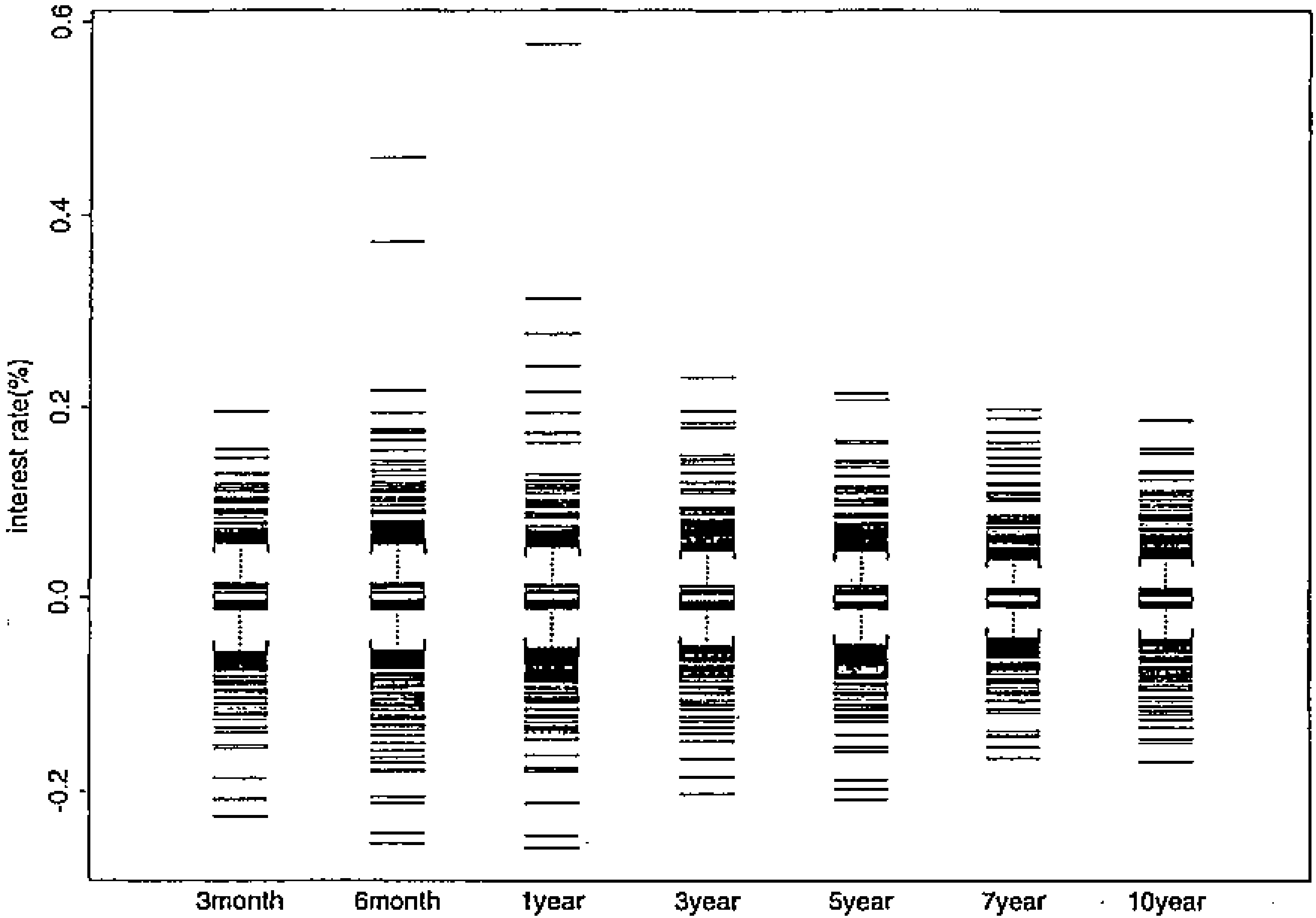


Figure 10. Boxplots of residuals

5. Prediction

Using the fitted MAR(2) model in Section 4.3, we can predict interest rates on a daily basis. For example, a practical one day ahead prediction of seven series is given at once by

$$\hat{Z}(t+1) = L(t) + S(t) + AI(t) + BI(t-1).$$

Table XI. Variability of each component of the decomposition

	3 months	6 months	1 year	3 years	5 years	7 years	10 years
long term trend	0.00716	0.00756	0.00827	0.00797	0.0078	0.00688	0.00709
short term trend	0.00518	0.00512	0.00499	0.00797	0.0078	0.00688	0.00709
irregular	0.07485	0.07611	0.07827	0.06976	0.0680	0.06345	0.05705

This is due to the fact that the variability of the irregular component dominates those of the long term trend and the short term trend as is seer in Table XI (see also Figure 5, 7 and 10). Here, the variabilities for the long and the short term are the standard deviations of the lag 1 differences when those were regarded as random fluctuations, and the variabilities for the irregular is exactly the standard deviations of the stationary process $I(t)$. Since the $Z(t+1)$ can be represented as

$$\hat{Z}(t+1) = L(t+1) + S(t+1) + AI(t) + BI(t-1) + \varepsilon(t+1),$$

the prediction error becomes

$$Z(t+1) - \hat{Z}(t+1) = \{L(t+1) - L(t)\} + \{S(t+1) - S(t)\} + \varepsilon(t+1).$$

The first two terms on the right hand side of the above equation are negligible compared with the last term $\varepsilon(t+1)$. The standard error of $\varepsilon(t+1)$ is around 0.03 to 0.04 as is seen in Table V, so that the prediction error is reduced to almost half by predicting with $\hat{Z}(t+1)$ based on the MAR(2) model, if it is compared with that by a simple prediction based only on the long term and short term trends,

$$\tilde{Z}(t+1) = L(t) + S(t).$$

Section 2. Hedge fund Returns

: A trial concerned with AR nature of Hedge fund returns.

Math Meth Oper Res (2009) 69:553–577
DOI 10.1007/s00186-008-0251-8

ORIGINAL ARTICLE

A note on statistical models for individual hedge fund returns

**Ryozo Miura · Yoshimitsu Aoki ·
Daisuke Yokouchi**

Received: 3 October 2007 / Accepted: 16 May 2008 / Published online: 10 September 2008

Outline

AR(+) months frequency is used in Part 1..

Then, monthly series of AR(+) months' AR(+) values are used in Part 2 .

:**************************************

1. Return data

: mean & Variances (moments) and AR(+) frequency.

2. (still going on)

2-1. (Variances. Adjusted R-squares)

Return regressions and Variability.

2-2. .(Means. Averages)

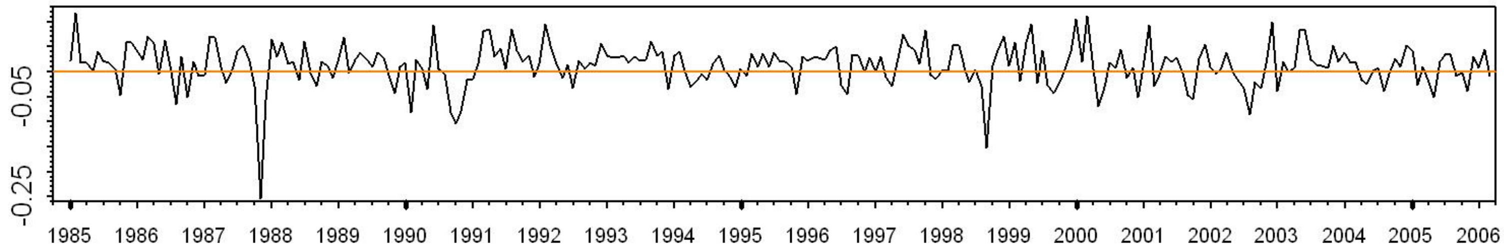
Return Regressions and Cross-sectional regressions for Intercepts and Averages.

Return data

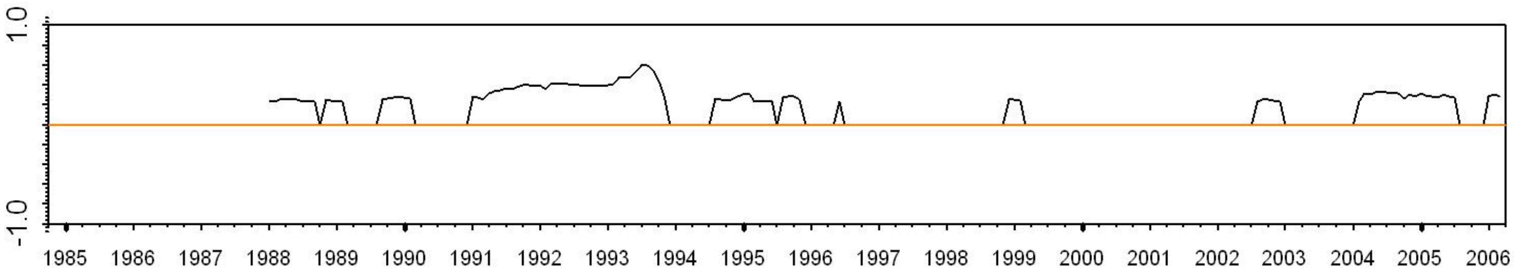
: mean & Variances (moments) and AR(+) frequency.

:

a) Return of a hedge fund



b) Coefficient of AR(1)



c) AIC(1) - AIC(0): AIC(n) is AIC for AR order n

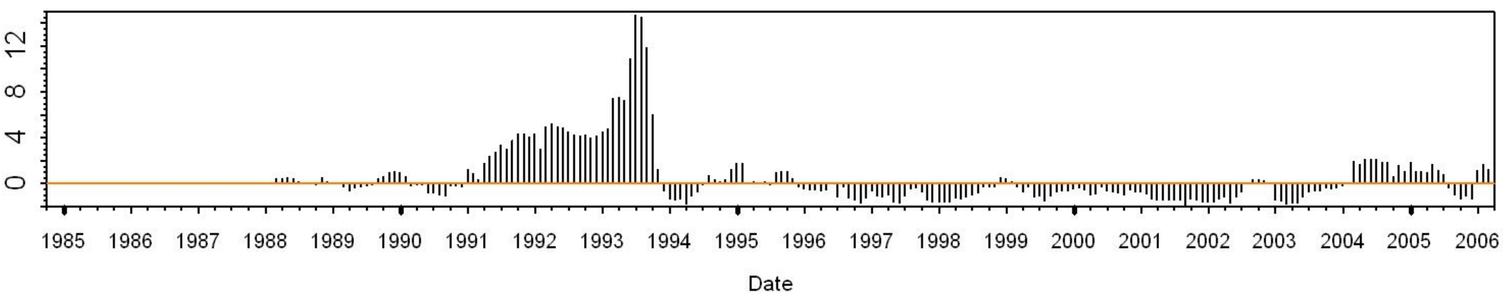
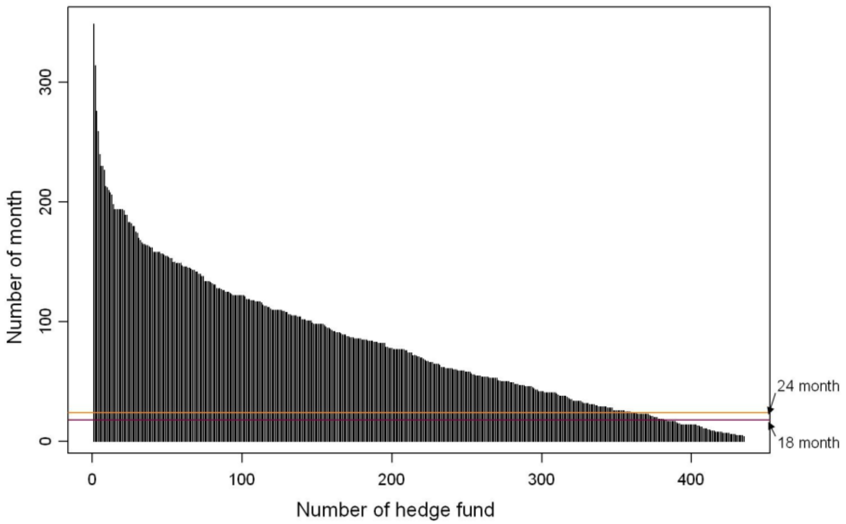


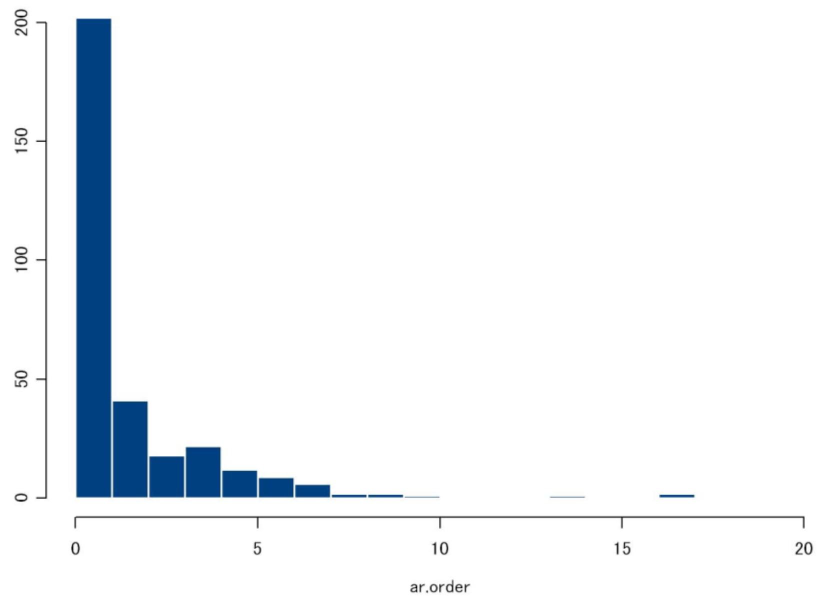
Fig .3 Typical structure of a hedge fund

Data : TASS-LIPPERS.

1984 Dec.=2006 Feb.

Data Length and Orders of AR :380 fund with 24 consecutive records





Reported Return Data

My Basic Worry/Wonder.

Return data time series is not an i.i.d. sequence, but it is Autoregressive.

The Mean-Variance approach is

(or implicitly thought of)

statistically based on

i.i.d. assumption?

AR(+) or AR(0).
We first identified each MONTH, either it is AR(+), or AR(0)

12 month data ending at the MONTH has AR structure of order 1 or more under AIC model selection, then it is AR(+). Other wise it is AR(0).

: Then, we see if the AR(+) frequency within 60 months, is related to "estimated" mean, variance, ("estimated" moments).

We identified all the month for all the individual hedge funds for analysis. This is one way to summarize , otherwise diversified results of statistical analysis.

Data （Lipper TASS）
218 Hedge Funds had 72 consecutive monthly records ending at February, 2006. We used this data.

Purpose of this work is, with the data taken from Lipper TASS ;

:Recognize the Autoregressive Structure of Hedge Fund (monthly)Returns

Statistical Data Analysis Has been done in the literatures implicitly assuming " Returns are Independent and Identically Distributed". We should like to see if those results are related to, or biased by its AR structure.

:Examine the following aspects that are usually said of Hedge Funds returns.
(1) Skew and kurtosis of distributions.
(2) Option features of returns
(3) some modifications for Sharpe Ratios

218 Funds in 10 categories

Category	The number of funds
Managed futures	47
Dedicated short bias	4
Multi-strategy	15
Equity market neutral	13
Fixed income arbitrage	3
Convertible arbitrage	15
Global macro	6
Event driven	19
Long/short equity hedge	76
Emerging markets	20

Mean & Variances vs. Frequency/60 of AR(+)

:a) Average Returns regressed on Freq. AR(+)

:Managed Futures

:b) Estimated standard deviations regressed on *

:Convertible Arbitrage

:c) Estimated skewness regressed on *

:Convertible Arbitrage

:d) Estimated kurtosis regressed on *

:Managed Futures

(Emerging Market has a strong positive slope)

:e) Option=Portfolio-like Nature by Hockey-Stick Regression

Convertible Arbitrage

:f) Sharpe Ratio

No clear relation in any category

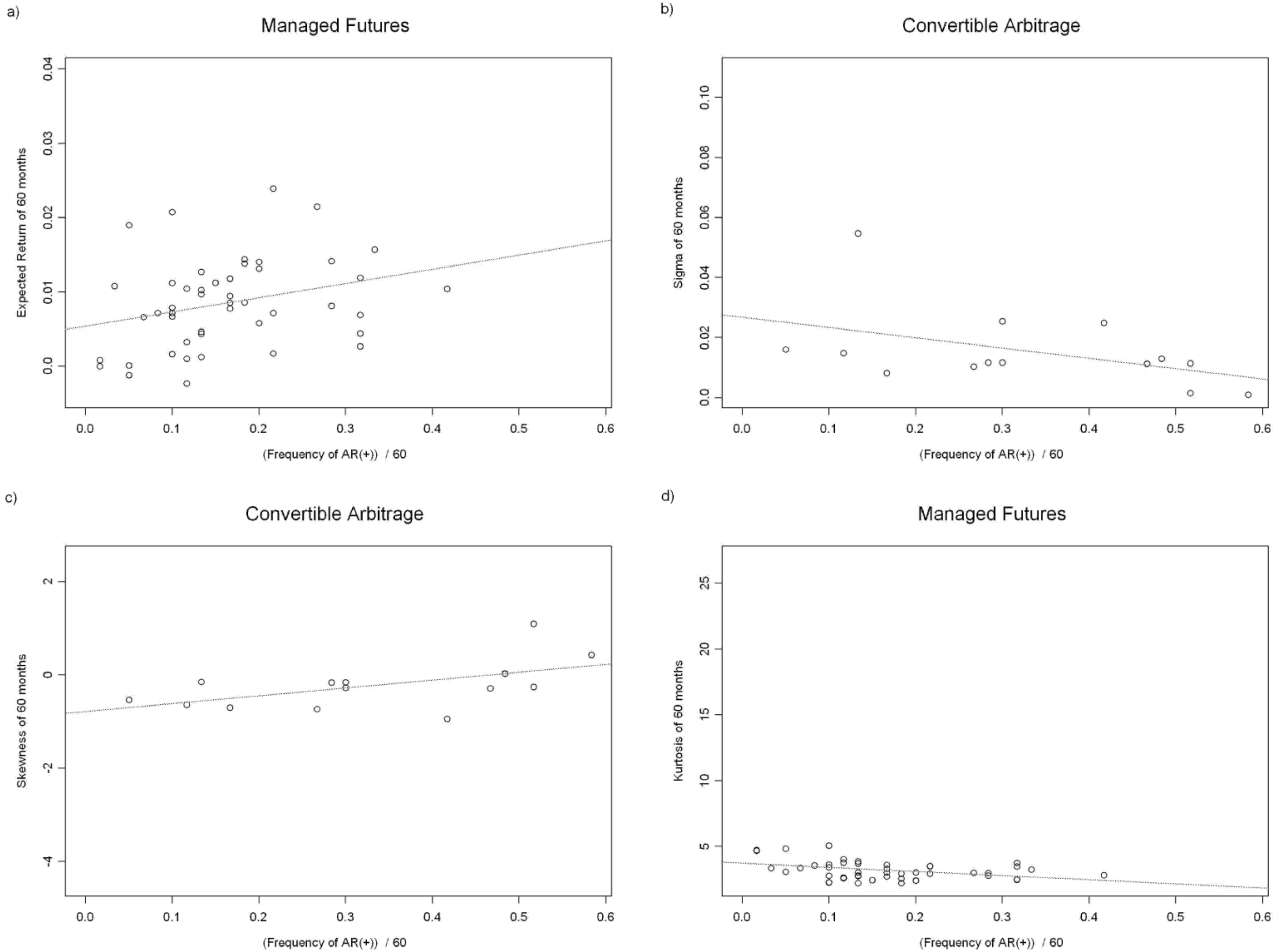


Fig. 7 a)Managed Futures: Average return regressed on the number of AR(+) months. b)Convertible Arbitrage: Estimated standard deviation regressed on the same. c)Convertible Arbitrage: Estimated skewness regressed on the same. d)Managed Futures: Estimated kurtosis regressed on the same

Section 3.

: A regression analysis to see how much they look like "Hedge-Funds".

: Trials for Data Analysis based on Brownian Nonparametric Statistics.

Section 3-1.

: A regression analysis to see how much they look like “Hedge-Funds”.

Independent regression variables (Factors in a style analysis)

f1： MSCI World INDEX Monthly returns

f2： MSCI Emerging Market Index Monthly return

f3： Citi World Government Bond Index   Monthly

f4： Citi Emerging Sovereign Bond Index   Monthly

PC1~PC5：Currency Exchange rate Principal Components from 9 currencies.

NY market

FH1~FH5： Look-back Straddles by Fung&Hsieh

Construct our Currency Factors

Currency Exchange Rates Factors

By Principal components

Based on 9 major currencies ;

(vs.US Dollar) JPYen, Euro, BPound,

SwissFranc, AustraliaDollar, CanadaDollar, NZDollar, Brazil, Chilli

為替のファクタを作成する。

分析対象期間を2002/08から2007/07の60ヶ月間とする（金融危機以前）

DATA: 2002.August===2007.July. 60months. Principal Components Analysis

2002/08から2007/07の間に月次変化率を算出できる以下の9通貨（対米ドル通貨ペア）

"円" "ﾕｰﾛ" "英ﾎﾟﾝﾄﾞ" "ｽｲｽ.ﾌﾗﾝ" "豪ﾄﾞﾙ" "ｶﾅﾀﾞ.ﾄﾞﾙ" "ﾆｭｰｼﾞｰﾗﾝﾄﾞﾄﾞﾙ" "ﾌﾞﾗｼﾞﾙﾚｱﾙ" "ﾁﾘ.ﾍﾟｿ "
JPYEN, EURO, BRPOUND, SWISS-FRANC, AUSYRALIA, CANADA, NEWJEALAND, BRAGIL, CHILLI

累積寄与度 Accumlated contribution

Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
0.421	0.740	0.815	0.874	0.921	0.959	0.982	0.997	1.000

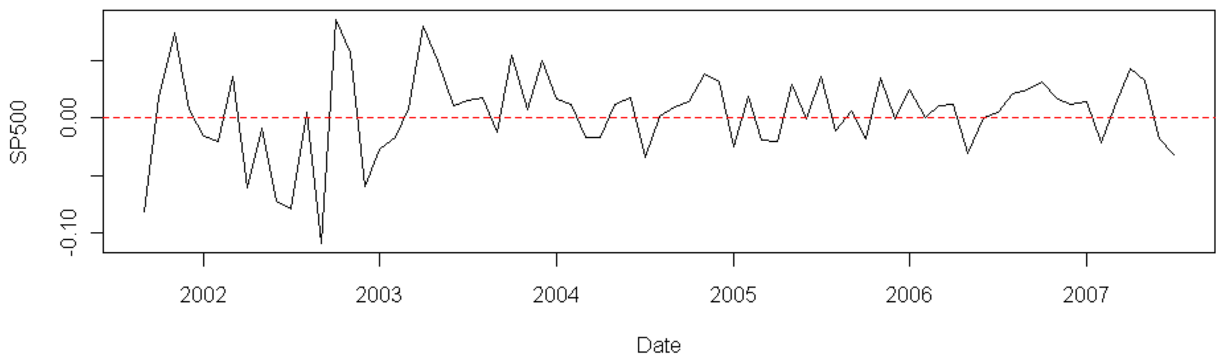
寄与度 Eigen Value Contribution

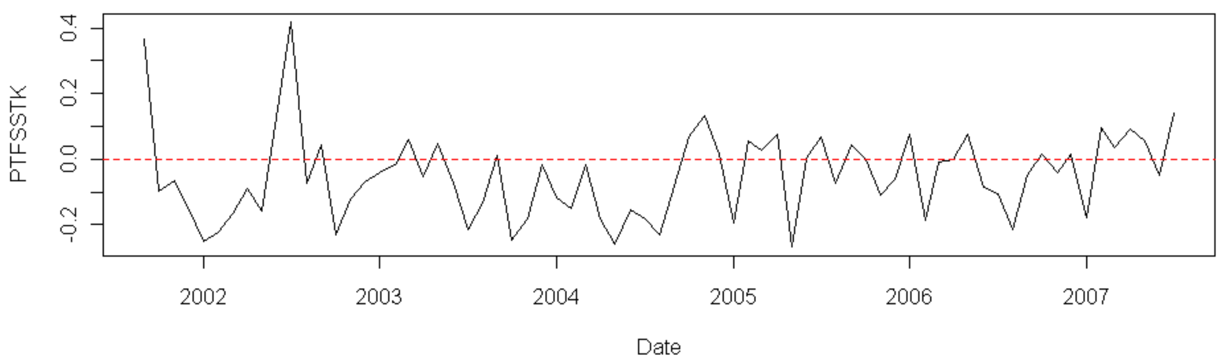
Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
0.421	0.319	0.076	0.059	0.047	0.038	0.023	0.016	0.003

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
円 JPYEN	0.146	0.225	-0.337	0.387	-0.160	0.773	0.106	0.107	0.141
ﾕｰﾛ EURO	-0.152	-0.394	0.302	-0.019	-0.055	0.368	-0.012	0.275	-0.718
英ﾎﾟﾝﾄﾞ Pound	-0.129	-0.359	0.104	0.131	-0.224	0.098	0.448	-0.752	0.036
ｽｲｽ.ﾌﾗﾝ SFRANC	0.101	0.449	-0.439	0.008	0.032	-0.230	0.278	-0.196	-0.653
豪ﾄﾞﾙAUSTRAL	-0.276	-0.302	-0.304	0.097	0.143	-0.196	0.641	0.487	0.158
ｶﾅﾀﾞ.ﾄﾞﾙCANADA	0.187	0.153	0.252	0.113	-0.838	-0.260	0.162	0.262	0.007
ﾆｭｰｼﾞｰﾗﾝﾄﾞﾄﾞﾙ	-0.215	-0.370	-0.498	0.411	-0.255	-0.234	-0.520	-0.052	-0.084
ﾌﾞﾗｼﾞﾙﾚｱﾙ BRAG	0.832	-0.460	-0.227	-0.201	0.027	-0.021	0.039	0.022	-0.019
ﾁﾘ.ﾍﾟｿ CHILI	0.281	0.026	0.369	0.775	0.363	-0.209	0.051	0.017	-0.069

赤枠部分をウェイトとしてとらえ、９通貨の月次変化率の系列と組み合わせることで、３本の通貨ファクタを作成した。
（単純に主成分得点を用いた場合には、計算の過程における中心化のため、平均が0となってしまう。
今回、ファクタの平均に着目した分析を行うために、上記のようなファクタ算出を行った）

Look-back Straddle (SP500) Return time series :Fung & Hsieh





These Replicated Look-back Straddle has negative average returns

We use these in our regressions to see the sensitivities of return variability with respect to these replicated Look-back Straddles.

i.e. we will do a cross-sectional regression later to analyze the average return of Hedge Funds.

回帰モデルの説明 Regression Models: Adding the independent variables

1．株式ファクタのみ、Stocks Factors Only
2．債券ファクタを追加（株式+債券）Stocks and Bonds Factors Only
3．為替ファクタを追加（株式+債券+為替）Stocks , Bonds, and Currencies
4．ヘッジファンドファクタを追加（株式+債券+為替+ヘッジファンドファクタ）Trend Follow Factors Added

model1 $$r = a + b_1 f_{先進株} + b_2 f_{途上株} + e$$

model2 $$r = a + b_1 f_{先進株} + b_2 f_{途上株} + b_3 f_{先進債} + b_4 f_{途上債} + e$$

model3 $$r = a + b_1 f_{先進株} + b_2 f_{途上株} + b_3 f_{先進債} + b_4 f_{途上債} + b_5 f_{為替1} + b_6 f_{為替2} + b_7 f_{為替3} + e$$

model4 $$r = a + b_1 f_{先進株} + b_2 f_{途上株} + b_3 f_{先進債} + b_4 f_{途上債} + b_5 f_{為替1} + b_6 f_{為替2} + b_7 f_{為替3} + b_8 f_{PTFSBD} + b_9 f_{PTFSFX} + b_{10} f_{PTFSCOM} + b_{11} f_{PTFSIR} + b_{12} f_{PTFSSTK} + e$$

$f_{先進株}$ MSCI World Index

$f_{途上株}$ MSCI Emerging Market Index

$f_{先進債}$ Citi World Goverment Bond Index

$f_{途上債}$ Citi Emerging Soverign Bond Index

$f_{為替1}$ 為替第１主成分 Currency 1st.Principal Cimponent

$f_{為替2}$ 為替第２主成分 Currency 2nd. PC

$f_{為替3}$ 為替第３主成分 Currency 3rd. PC

f_{PTFSBD}

f_{PTFSFX}

$f_{PTFSCOM}$

f_{PTFSIR}

$f_{PTFSSTK}$

Decomposition of Returns into three components.

$$R_t = \mu_t + \sum_{i=1}^{k_t} a_{i,t}(R_{t-i} - \mu_t) + \varepsilon_t$$

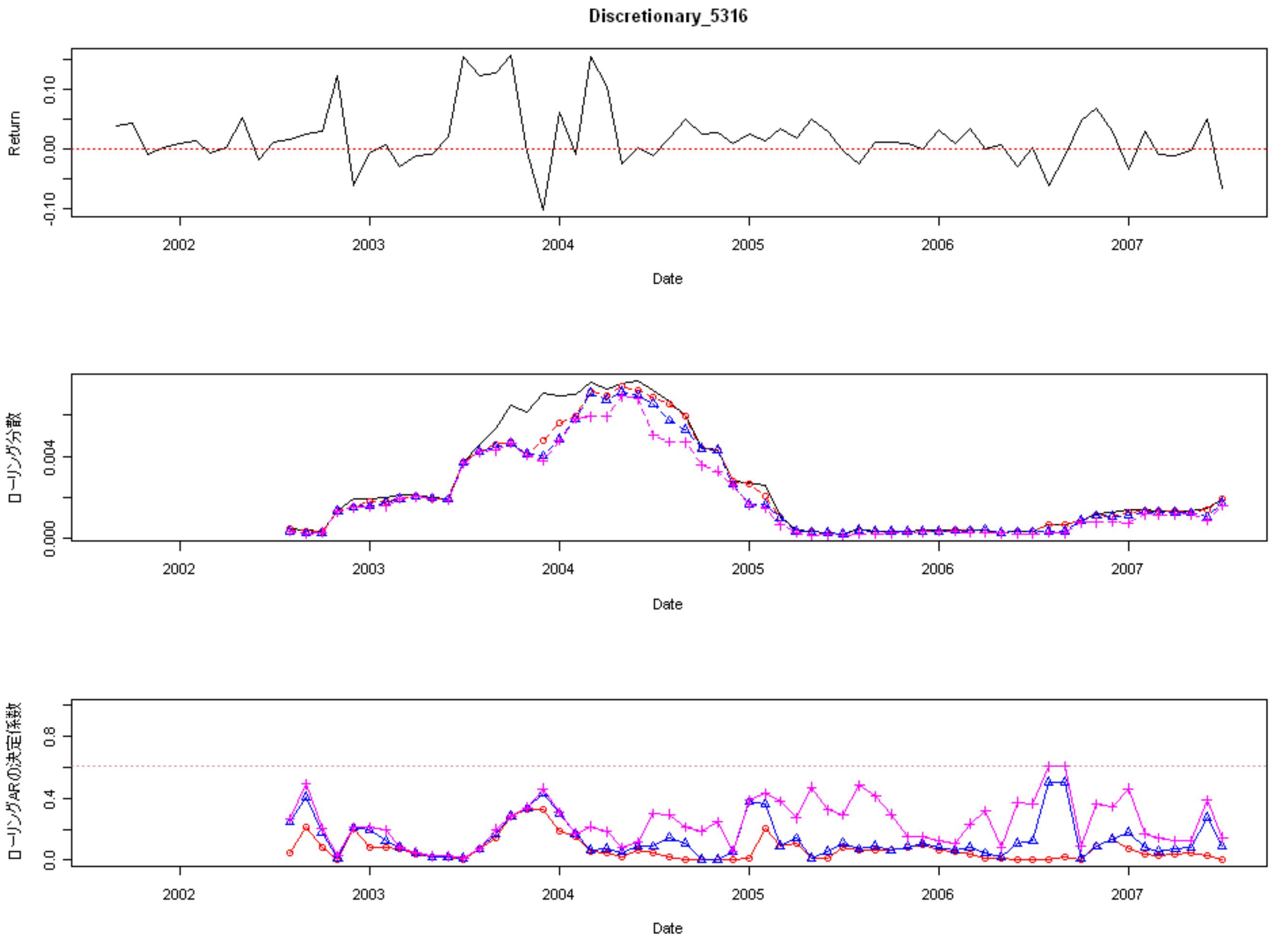
Not only the whole returns, but also the AR part will be regressed on the factors. To see how it will be explained.

Our View on AR part of Returns
Hedge Fund managers/traders may well know AR part of the month (?) since they know what they are doing. Then, the stochastics to them is not Rt, but only in ε.
Then, investors outside would not know what AR for the month be.

Variability

: AR structure occupies quite a part of return variance.

:So, we look at both Whole return and its AR part.



Emerging Markets : 55 funds

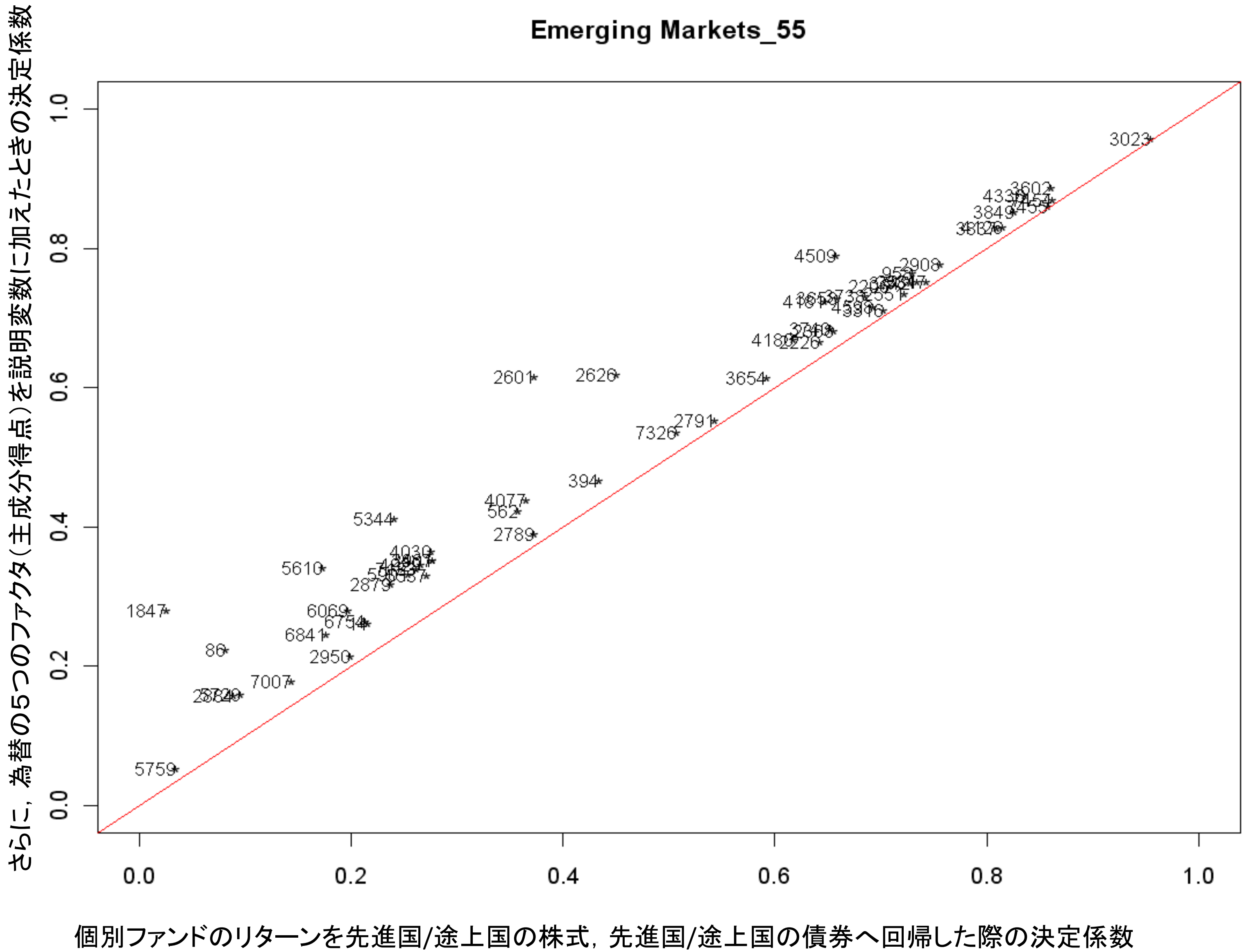
We look at how much the fitting is improved with the additional groups of factors.
Then, we look at if it is related to the levels of Average Returns

: (1) Each factor group provides improved Fitting (Adjusted R-square).

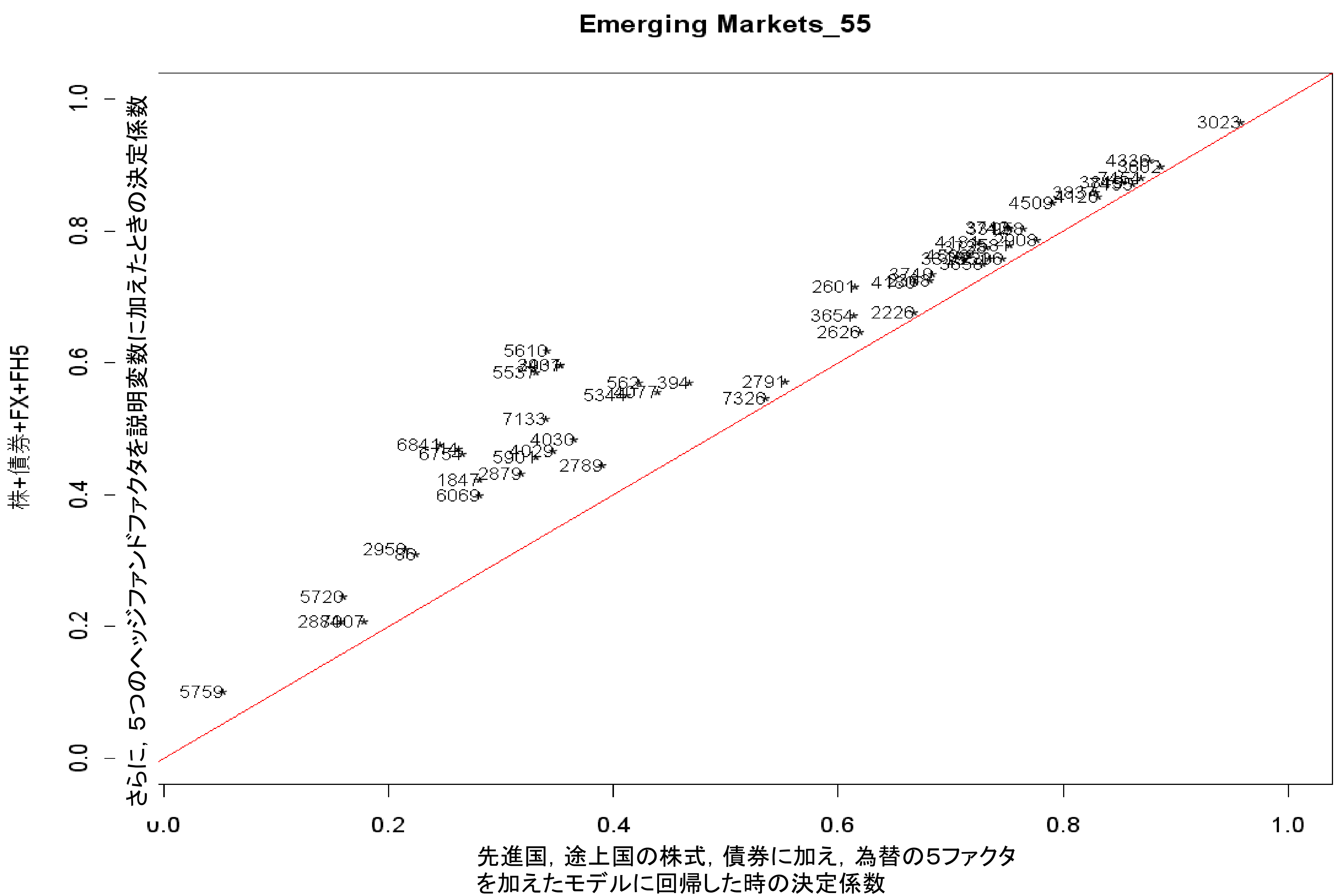
(This means that Hedge Funds have common factors with a usual [Stock+Bond] portfolio; This may be consistent with a statement; "Including Hedge Funds into a usual portfolio helps to increase its RETURN and decrease its RISK, but only to certain percentage amount. If it takes more than that, it will increase RISK".)

****

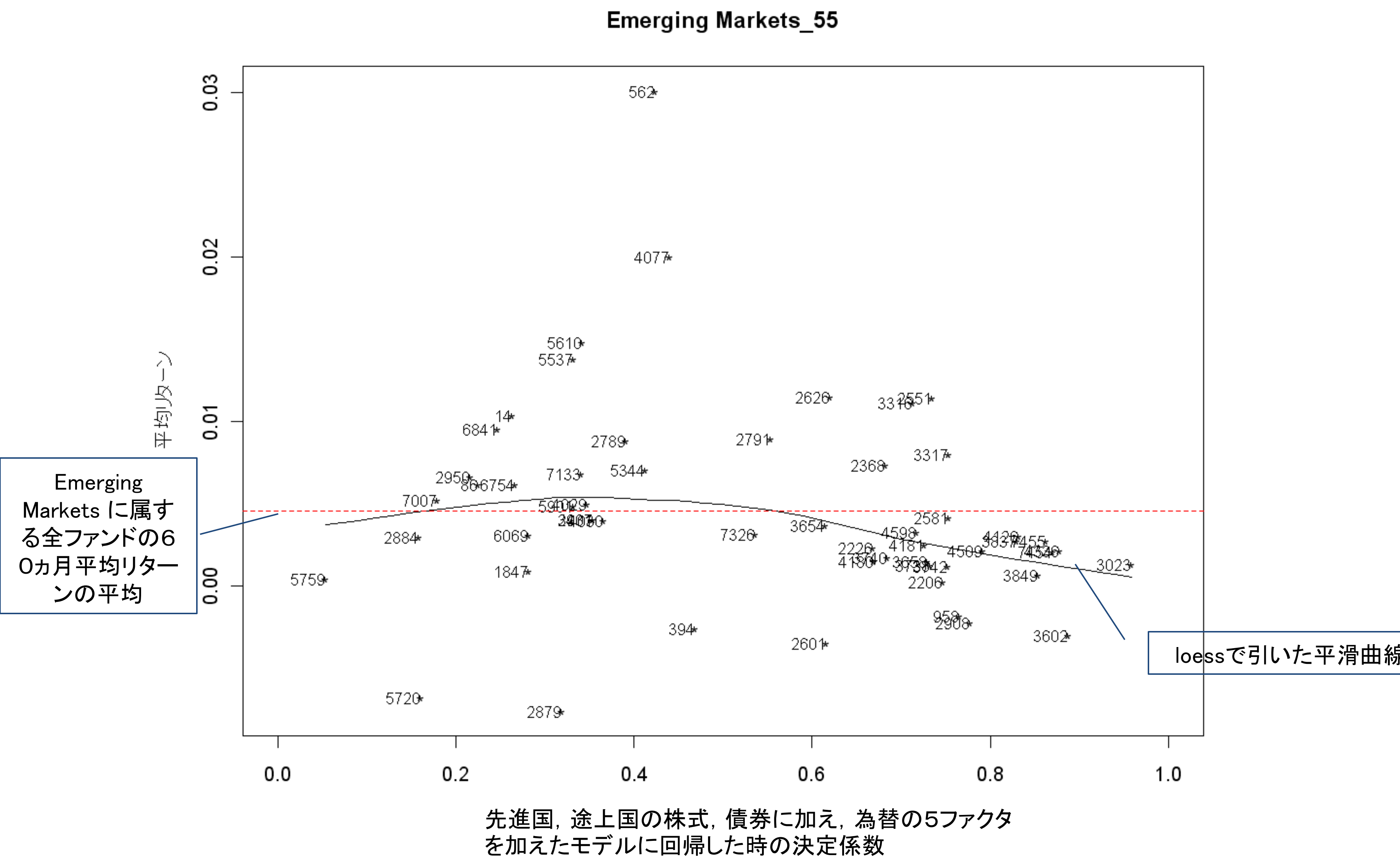
: (2) 41 funds out of 55 funds in this category have their Adjusted R-squares improved with look-back straddles.



Emerging Markets_55
さらに，為替の5つのファクタ（主成分得点）を説明変数に加えたときの決定係数
個別ファンドのリターンを先進国/途上国の株式，先進国/途上国の債券へ回帰した際の決定係数
0.0
0.2
0.4
0.6
0.8
1.0



Emerging Markets_55
株+債券+FX+FH5
さらに，5つのヘッジファンドファクタを説明変数に加えたときの決定係数
先進国，途上国の株式，債券に加え，為替の5ファクタ
を加えたモデルに回帰した時の決定係数
0.0
0.2
0.4
0.6
0.8
1.0
3023
4509
2601
3654
2226
2626
5610
5537
2791
7326
394
7133
4030
2789
2879
1847
6069
5901
2959
5720
5759



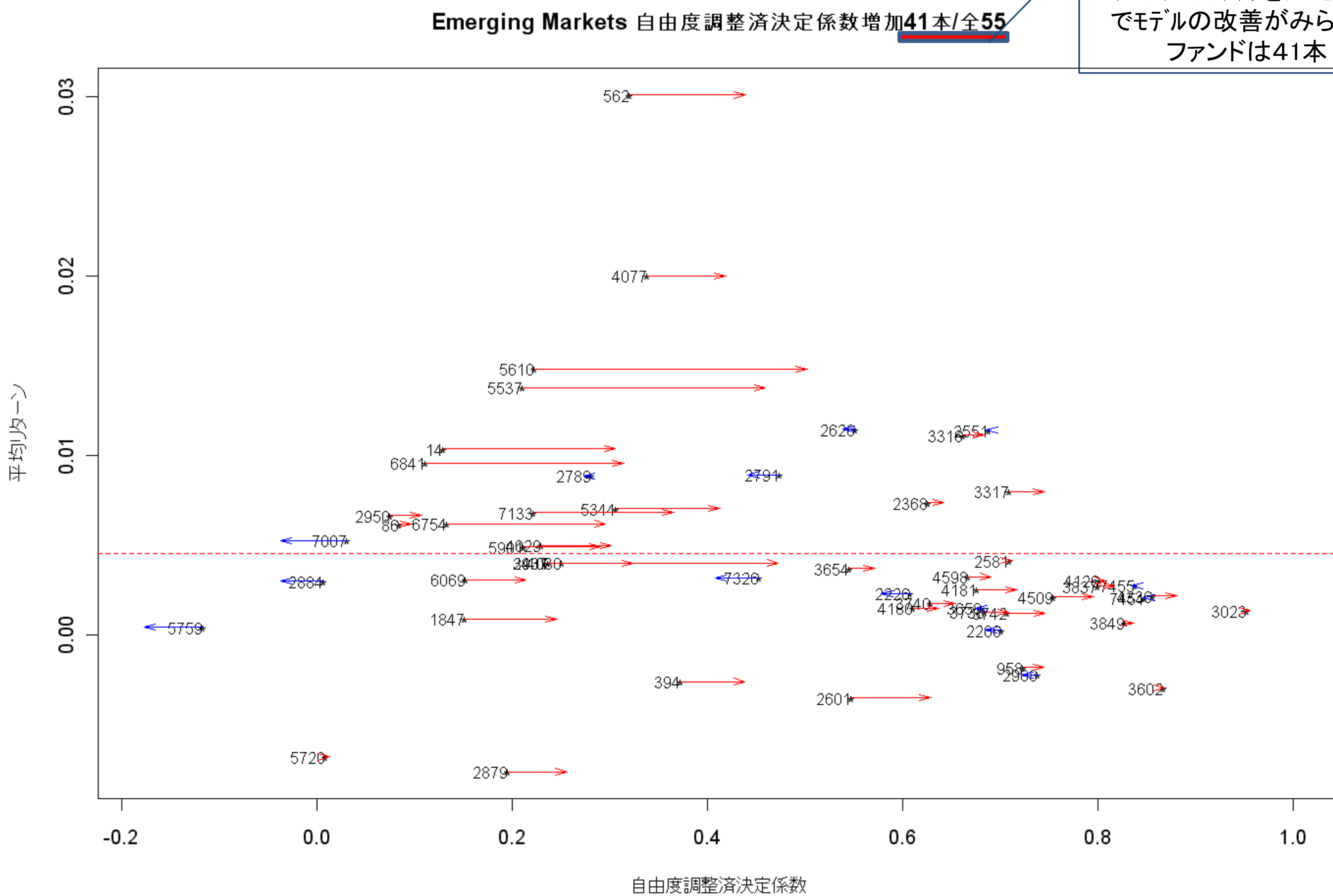
Emerging Markets_55
平均リターン
0.03
0.02
0.01
0.00
0.0
0.2
0.4
0.6
0.8
1.0
Emerging Markets に属する全ファンドの60ヵ月平均リターンの平均
loessで引いた平滑曲線
先進国，途上国の株式，債券に加え，為替の5ファクタを加えたモデルに回帰した時の決定係数

自由度調整済決定係数による比較を行うことで，ﾍｯｼﾞﾌｧﾝﾄﾞﾌｧｸﾀを加えたことがﾓﾃﾞﾙの改善に寄与しているか否かを検証する．

ﾍｯｼﾞﾌｧﾝﾄﾞﾌｧｸﾀを加えたことによって，自由度調整済決定係数が上昇した場合，**右向きの赤矢印**で表現

自由度調整済決定係数が減少した場合には**左向き青矢印**で表現

Emerging Markets に属する55本のファンドのうち，ﾍｯｼﾞﾌｧﾝﾄﾞﾌｧｸﾀを加えたことでﾓﾃﾞﾙの改善がみられたファンドは41本



Next, we check if AR structure remains in the residuals.

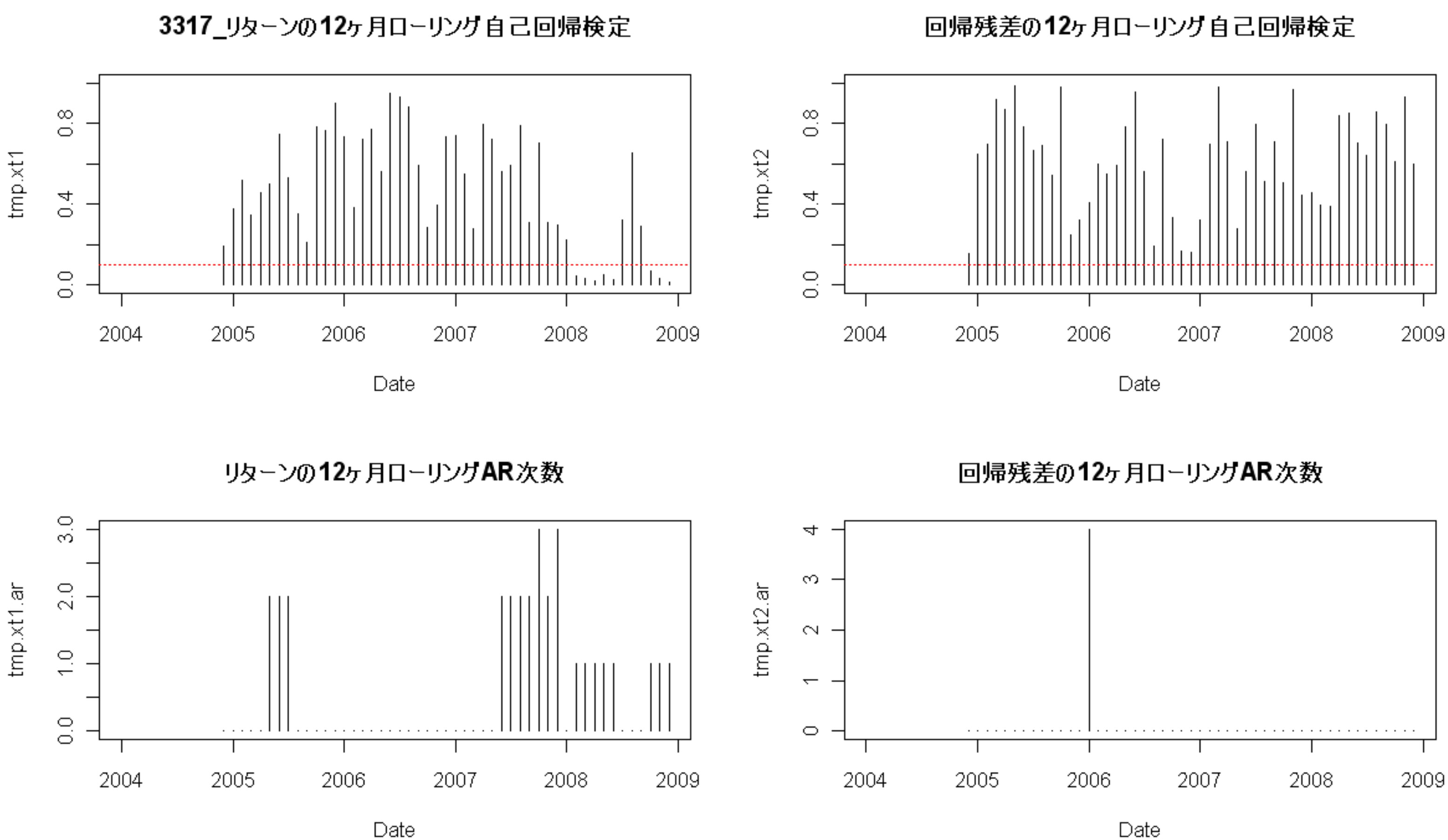
The results

:The factors we used here do not clearly take off, or add AR structures either. They do not seem to matter in this regard.

Some funds reduces AR(+) frequency in the residuals, some others do not, and some even increase; As shown below.

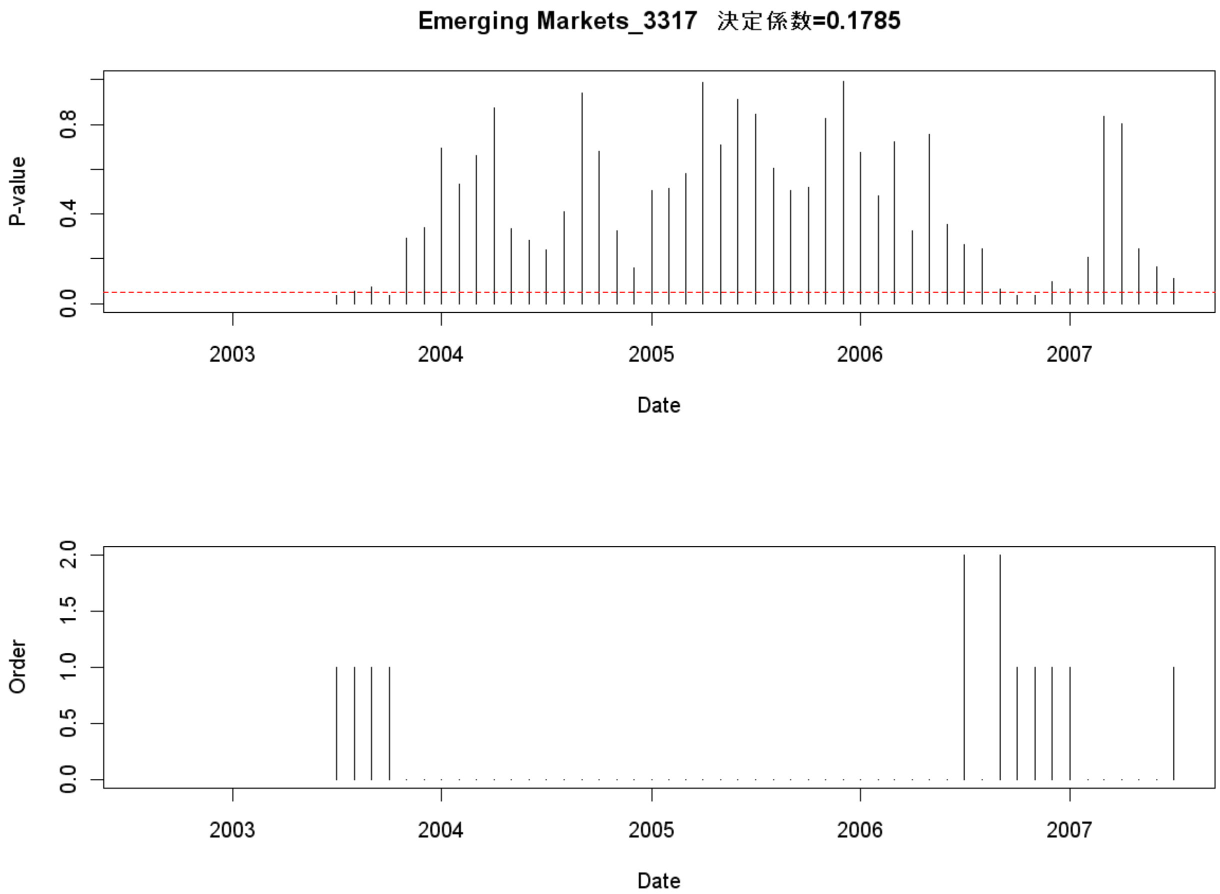
ファンドのリターン系列，「株，債券，為替のﾌｧｸﾀﾓﾃﾞﾙに回帰した時の残差」，
それぞれに対し，１２カ月ローリングで自己回帰性の検定（Ljung-Box検定）のP値，
１２ヵ月ローリングでのARモデル当てはめの次数（AICによる次数選択）を行った結果

Upper figure : P-value for testing Hypothesis “No Correlation” .　Lower :Order of AR by AIC



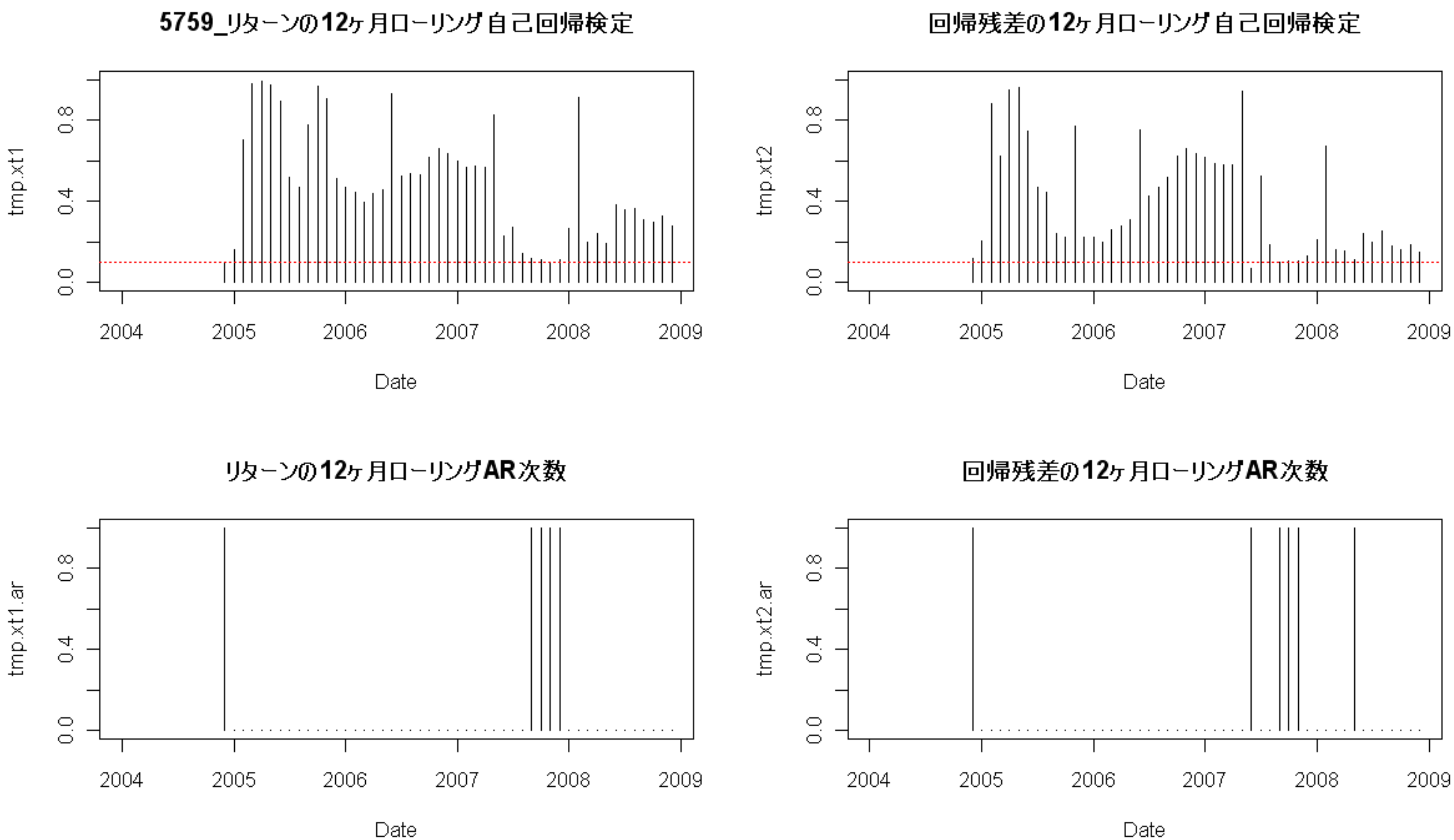
AR in residuals in the last regression model. 3317

Upper figure : P-value for testing Hypothesis “No Correlation” .   Lower :Order of AR by AIC



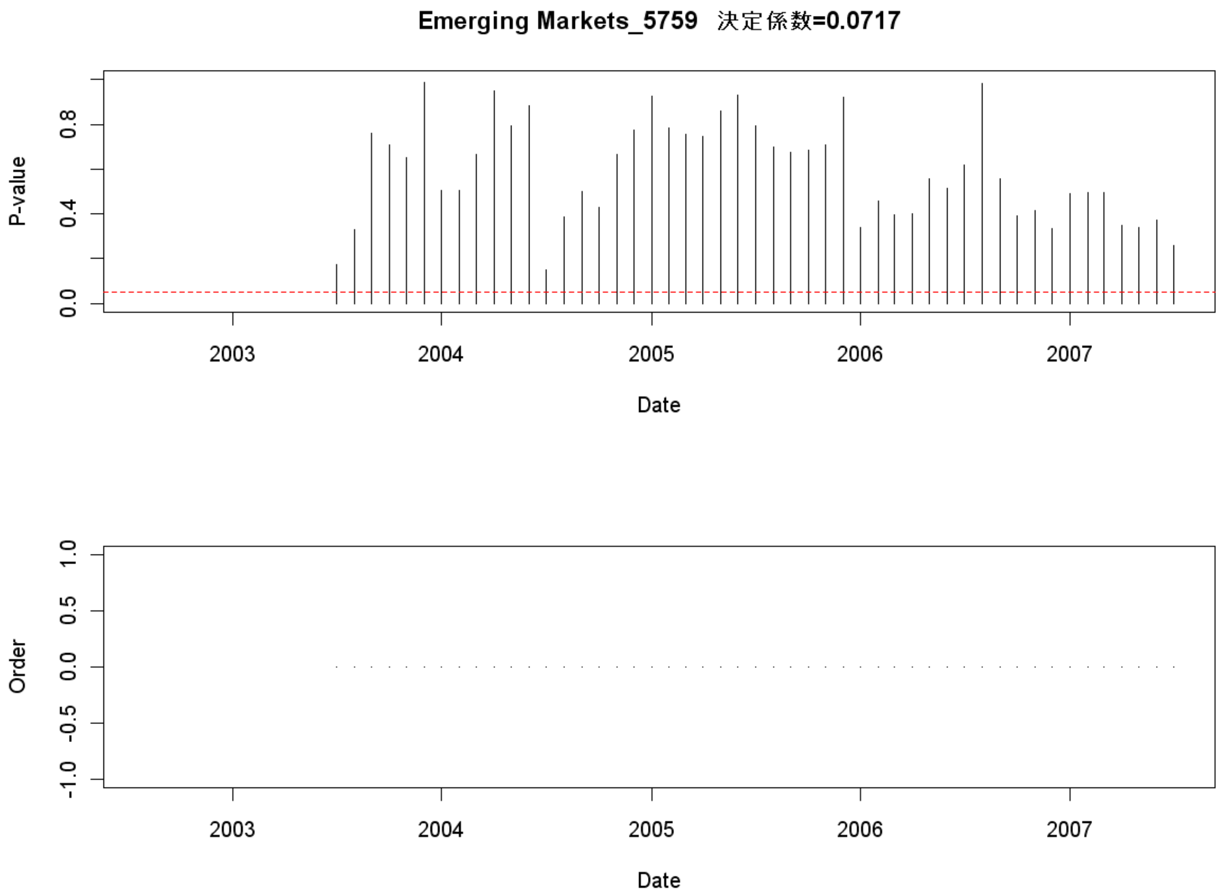
ファンドのリターン系列，「株，債券，為替のﾌｧｸﾀﾓﾃﾞﾙに回帰した時の残差」，
それぞれに対し，12カ月ローリングで自己回帰性の検定（Ljung-Box検定）のP値，
12ヵ月ローリングでのARモデル当てはめの次数（AICによる次数選択）を行った結果

Upper figure : P-value for testing Hypothesis “No Correlation” .　Lower :Order of AR by AIC

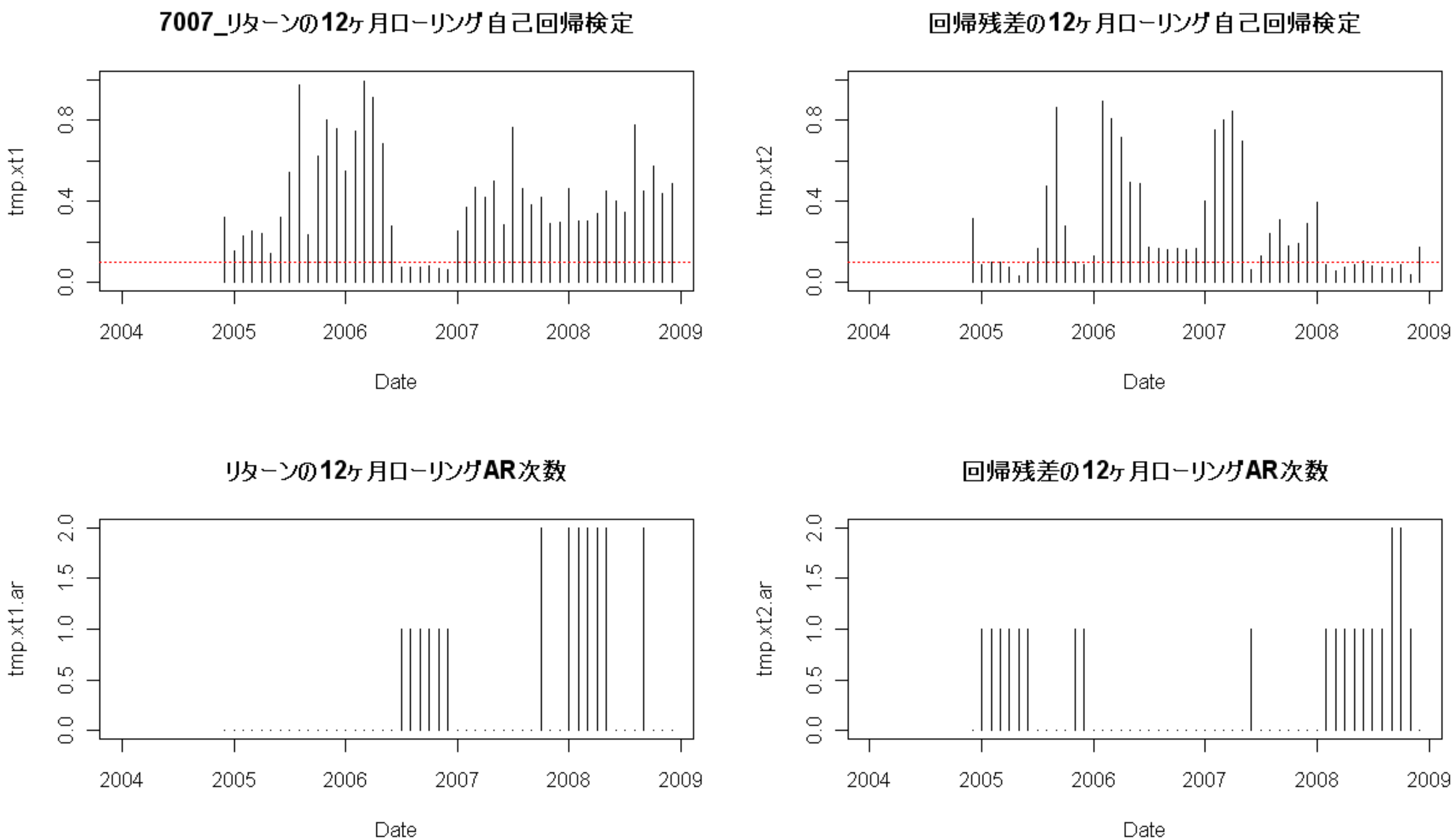


AR in residuals in the last regression model. 5759

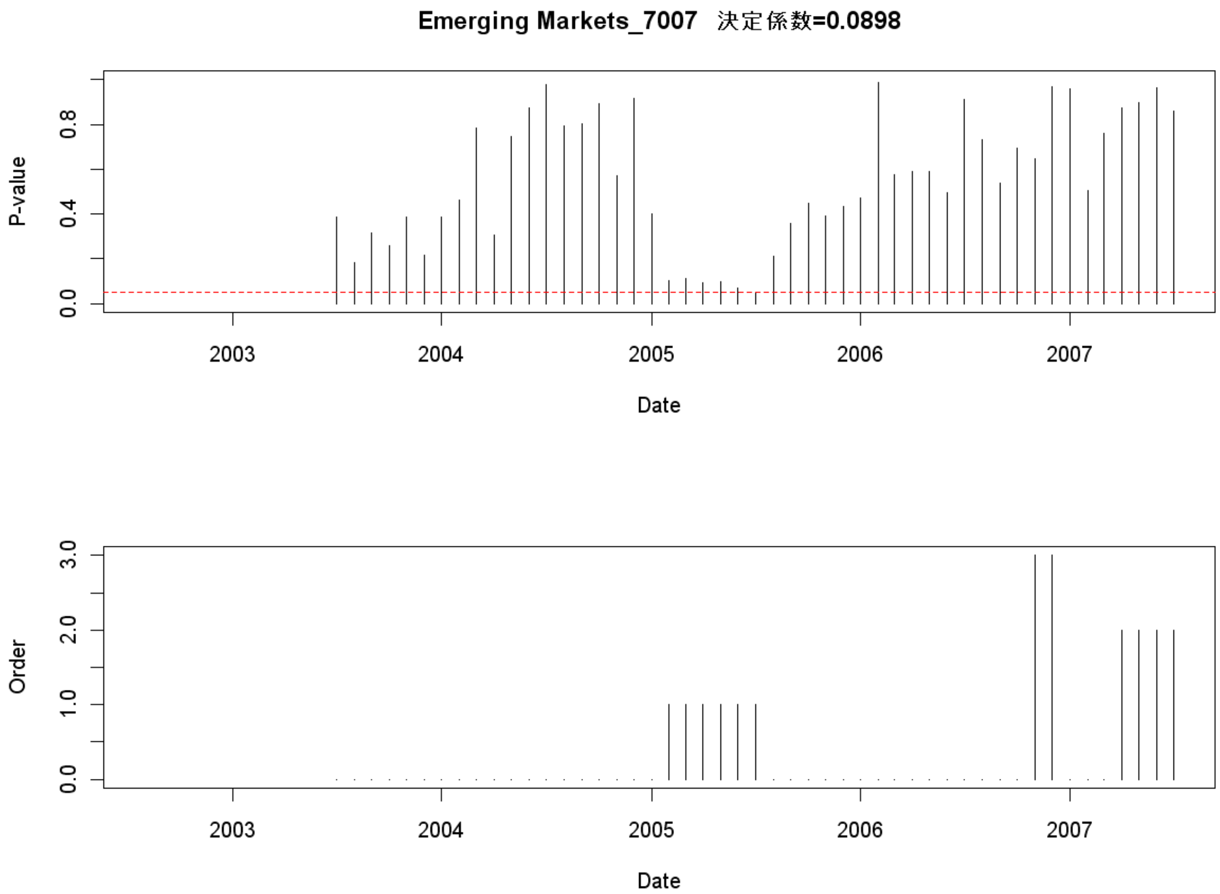
Upper figure : P-value for testing Hypothesis “No Correlation” . Lower :Order of AR by AIC



ファンドのリターン系列，「株，債券，為替のﾌｧｸﾀﾓﾃﾞﾙに回帰した時の残差」，
それぞれに対し，１２カ月ローリングで自己回帰性の検定（Ljung-Box検定）のP値，
１２ヵ月ローリングでのARモデル当てはめの次数（AICによる次数選択）を行った結果

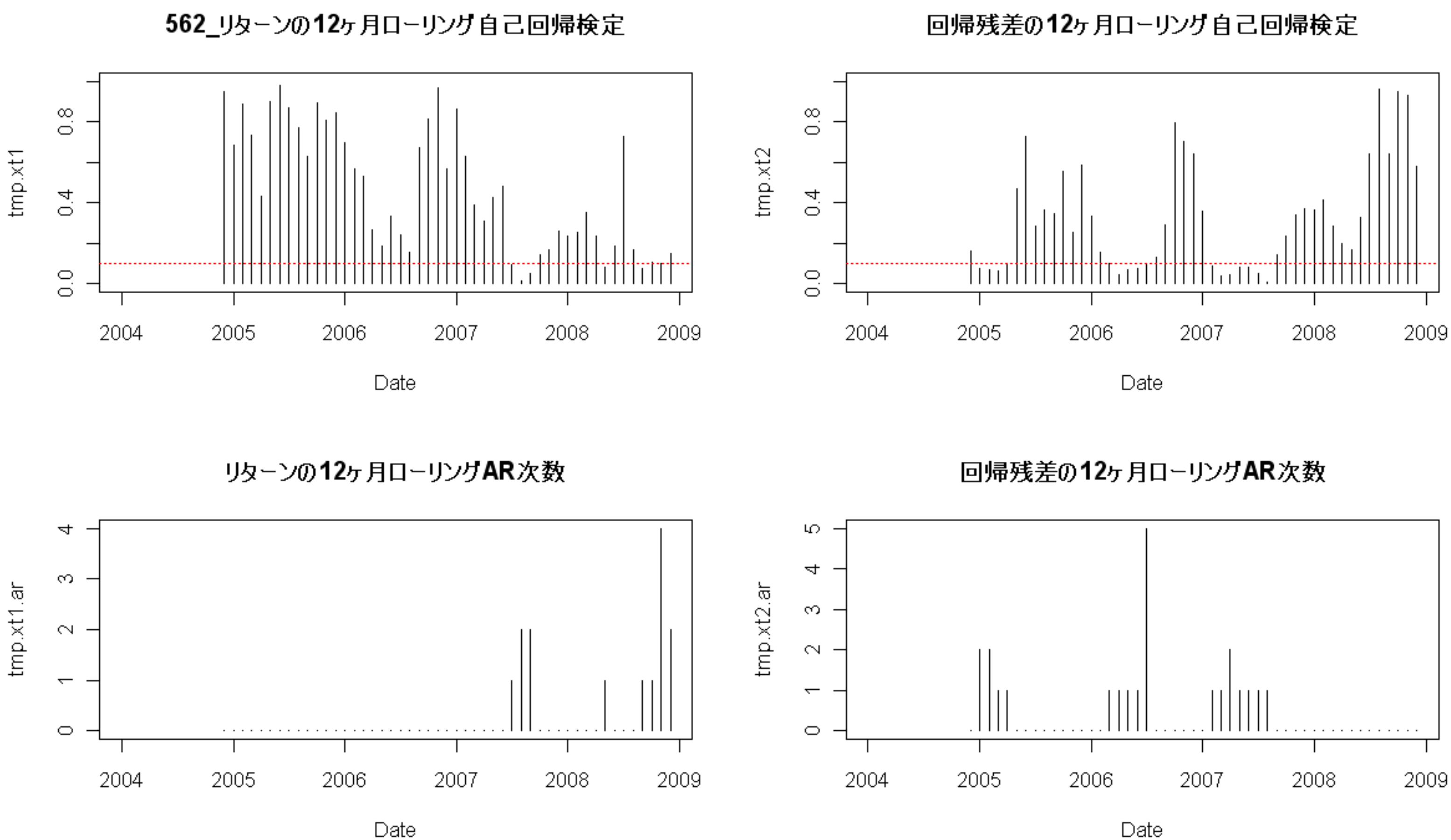


AR in residuals in the last regression model. 7007



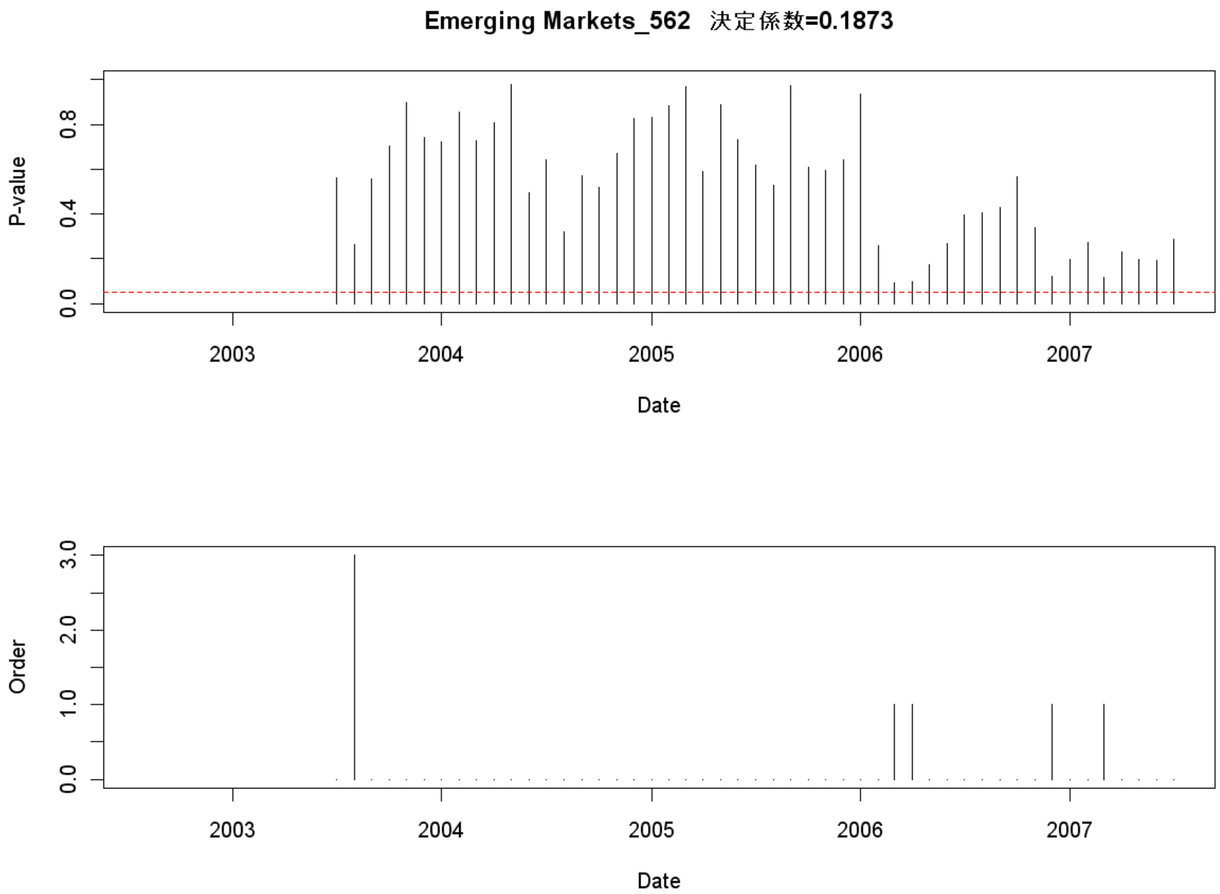
ファンドのリターン系列，「株，債券，為替のﾌｧｸﾀﾓﾃﾞﾙに回帰した時の残差」，
それぞれに対し，12カ月ローリングで自己回帰性の検定（Ljung-Box検定）のP値，
12ヵ月ローリングでのARモデル当てはめの次数（AICによる次数選択）を行った結果

Upper figure : P-value for testing Hypothesis “No Correlation”． Lower :Order of AR by AIC



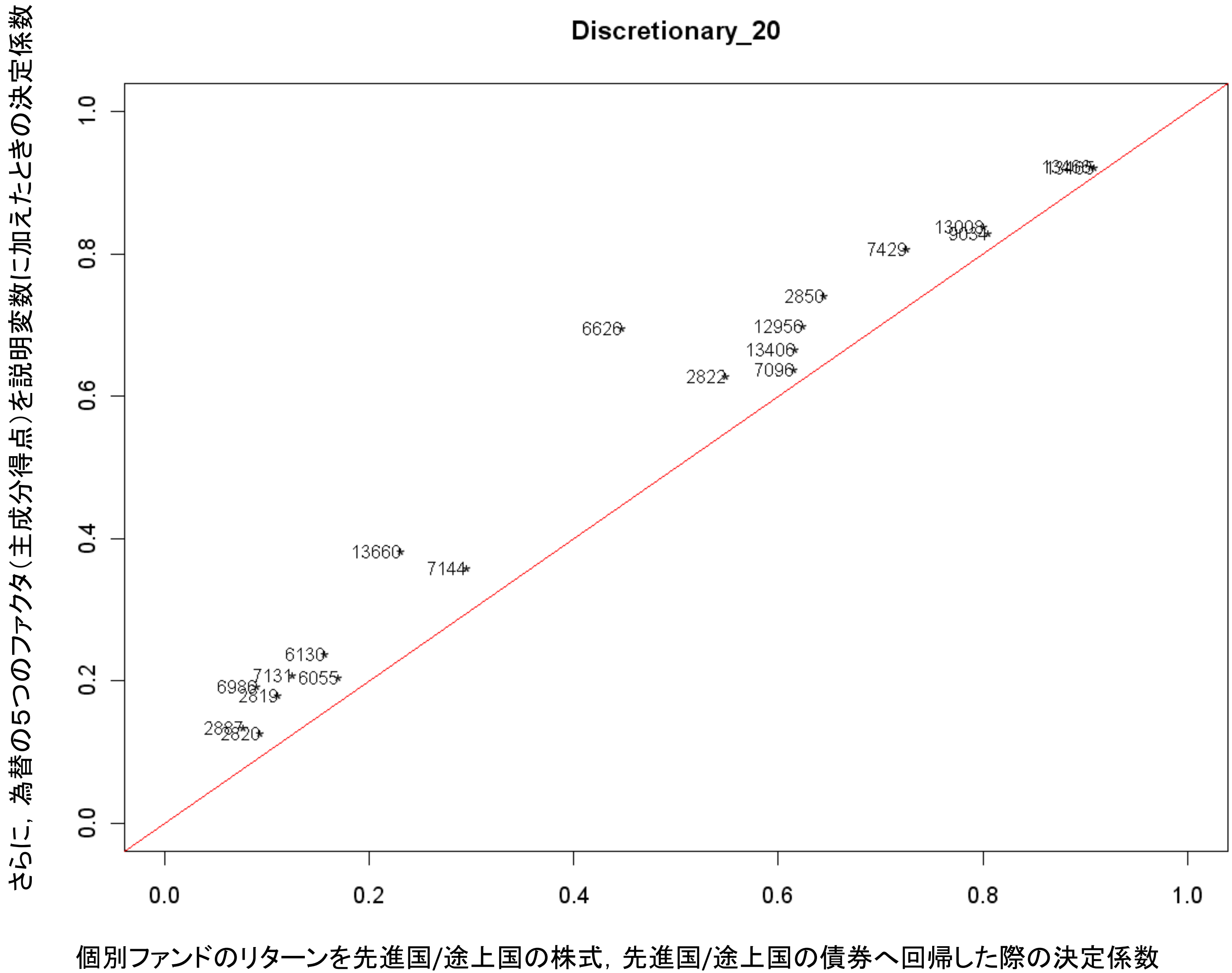
AR in residuals in the last regression model. 562

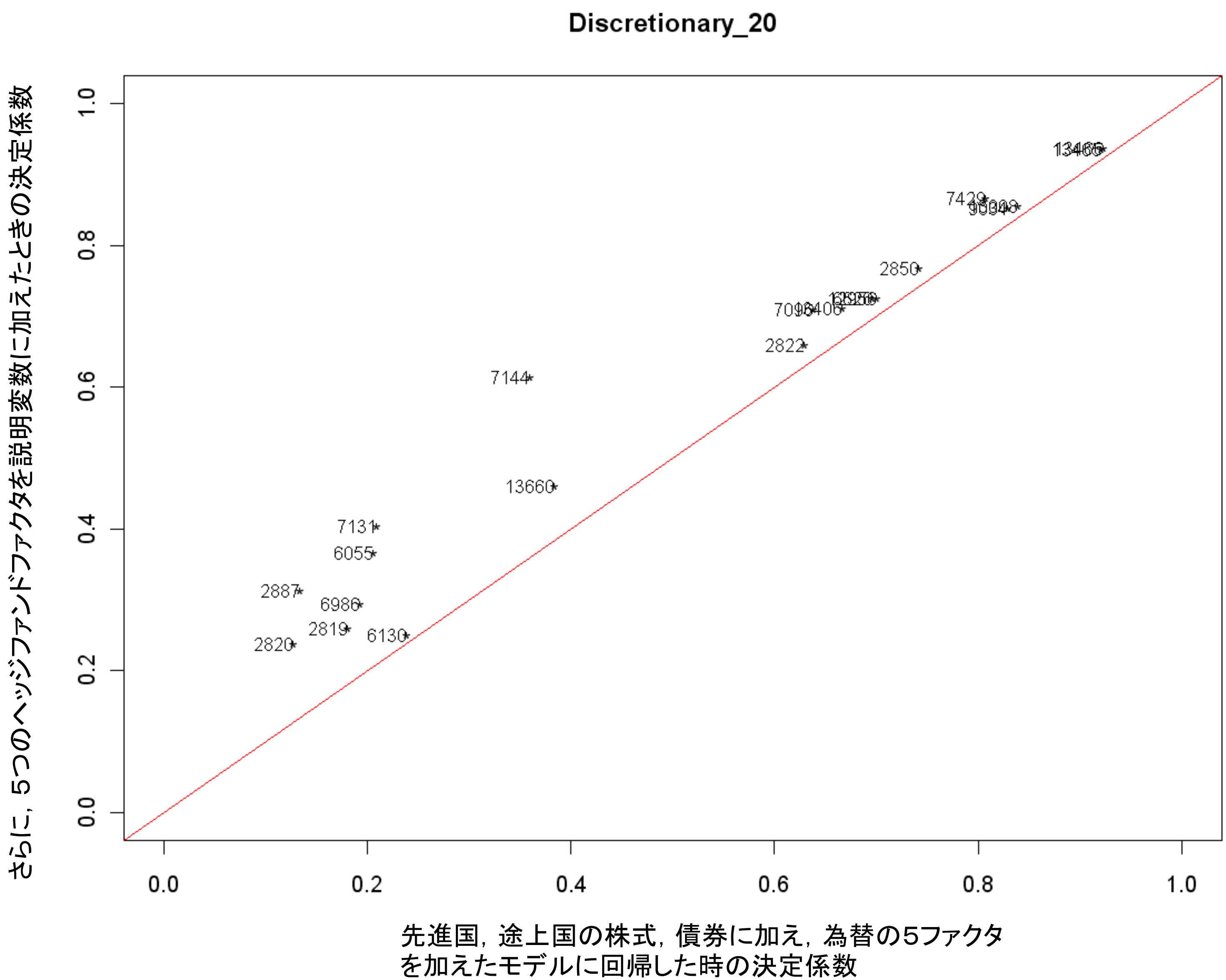
Upper figure : P-value for testing Hypothesis “No Correlation” .   Lower :Order of AR by AIC



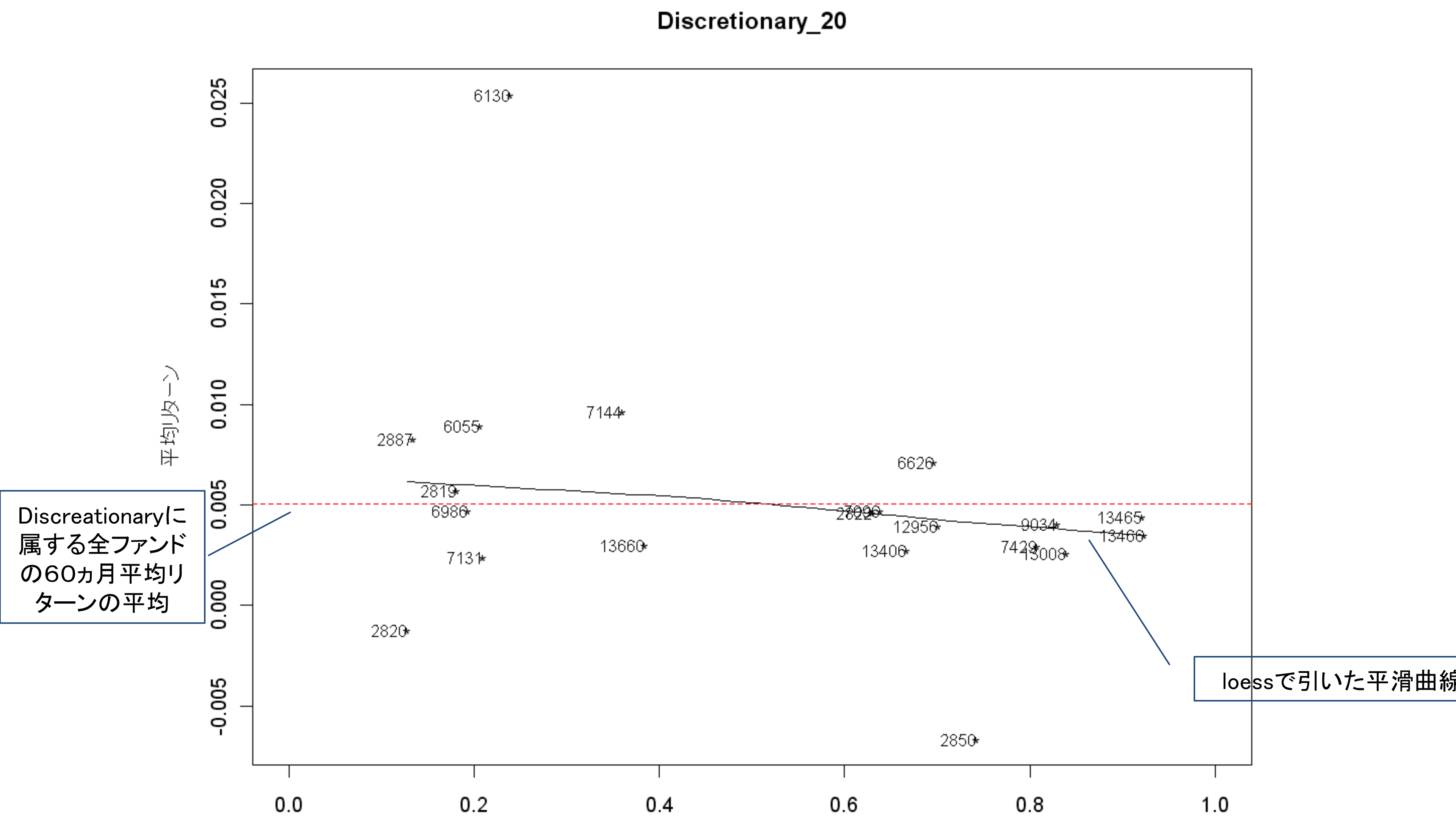
Discretionary

:


Discretionary_20
さらに，為替の5つのファクタ（主成分得点）を説明変数に加えたときの決定係数
個別ファンドのリターンを先進国/途上国の株式，先進国/途上国の債券へ回帰した際の決定係数
0.0
0.2
0.4
0.6
0.8
1.0
13465
13009
9034
7429
2850
6626
12956
13406
2822
7096
13660
7144
6130
7131
6055
6986
2819
2887
2820



Discretionary_20
さらに，5つのヘッジファンドファクタを説明変数に加えたときの決定係数
先進国，途上国の株式，債券に加え，為替の5ファクタを加えたモデルに回帰した時の決定係数
0.0
0.2
0.4
0.6
0.8
1.0
13465
7429
2850
2822
7144
13660
7131
6055
2887
6986
2819
6130
2820

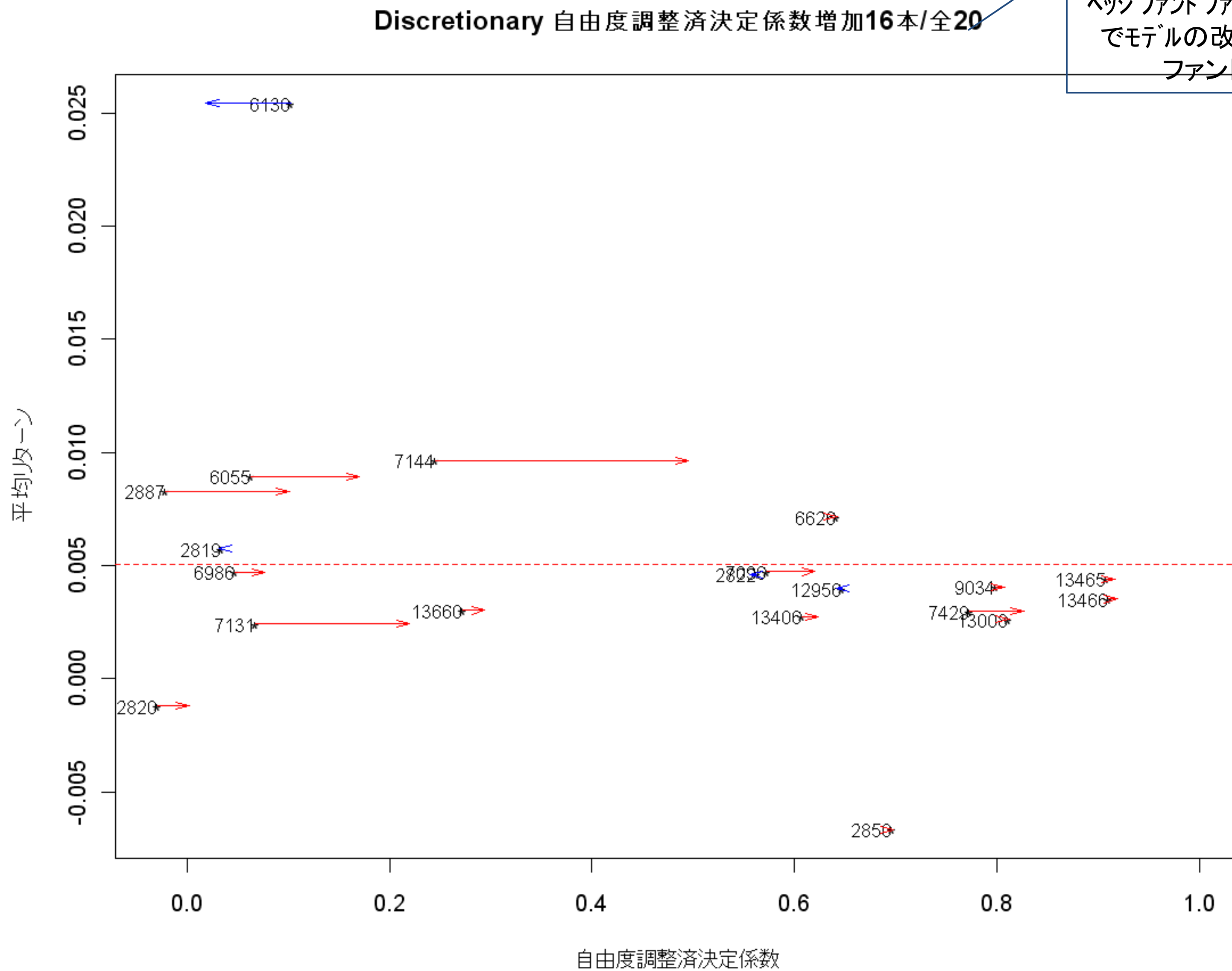


先進国，途上国の株式，債券に加え，為替の5ファクタを加えたモデルに回帰した時の決定係数

自由度調整済決定係数による比較を行うことで，ﾍｯｼﾞﾌｧﾝﾄﾞﾌｧｸﾀを加えたことがﾓﾃﾞﾙの改善に寄与しているか否かを検証する.

ﾍｯｼﾞﾌｧﾝﾄﾞﾌｧｸﾀを加えたことによって，自由度調整済決定係数が上昇した場合，**右向きの赤矢印**で表現
自由度調整済決定係数が減少した場合には**左向き青矢印**で表現

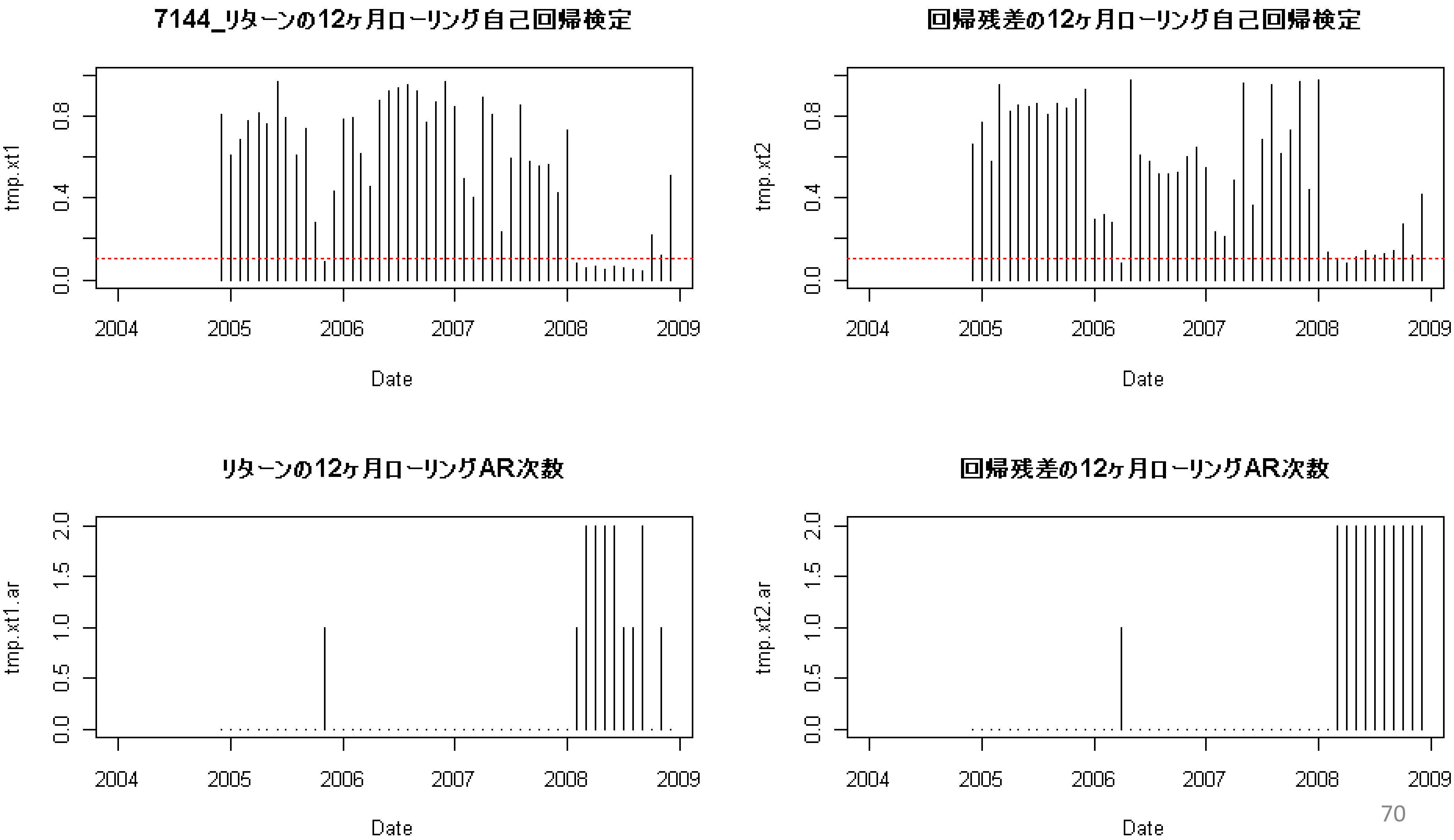
Discreationaryに属する20本のファンドのうち，ﾍｯｼﾞﾌｧﾝﾄﾞﾌｧｸﾀを加えたことでﾓﾃﾞﾙの改善がみられたファンドは16本



We should like to know if look-back straddle reduces the AR structure(AR(+) freuency) of returns since this added factor is expected to explain a nature of Hedge Funds.

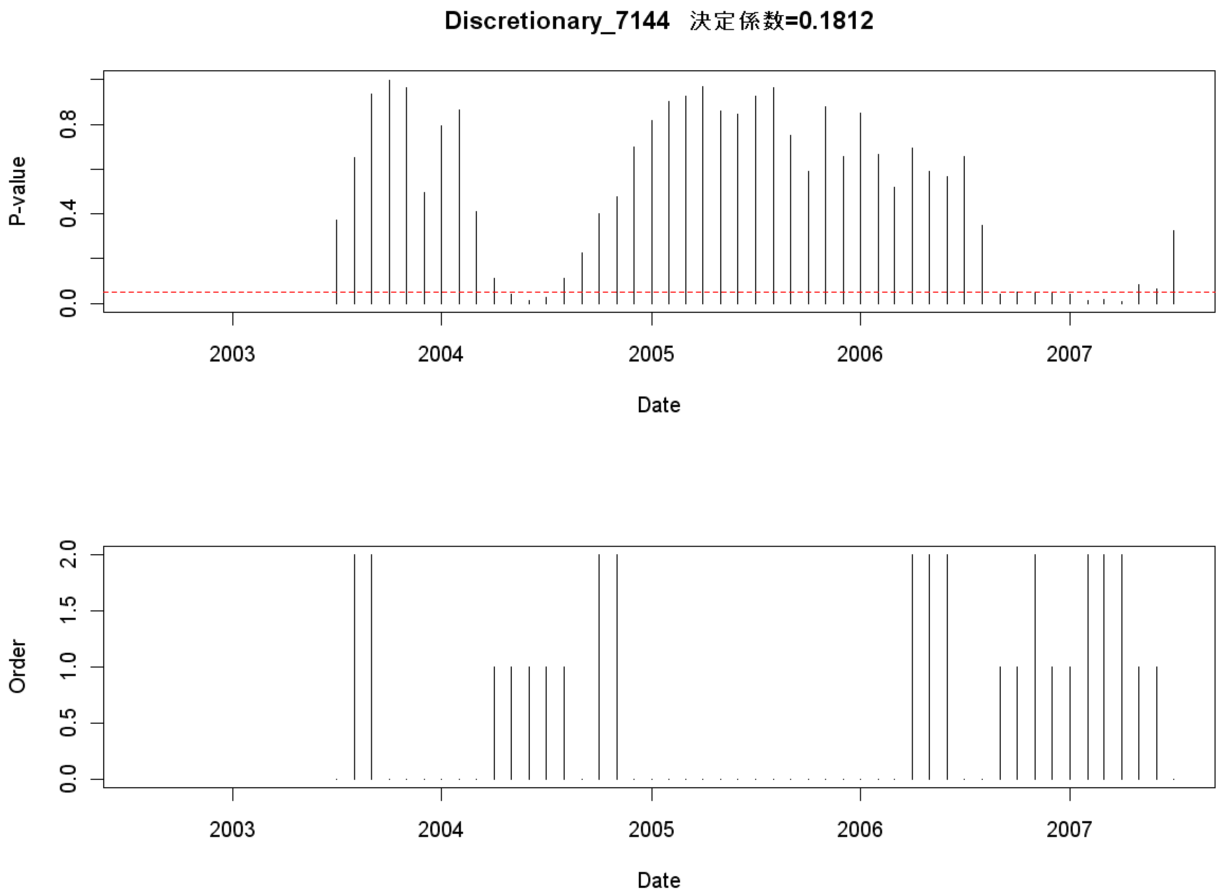
:The result here shows “Not Really So” with this factor (only?). The regression residuals still have AR structures.

ファンドのリターン系列，「株，債券，為替のﾌｧｸﾀﾓﾃﾞﾙに回帰した時の残差」，
それぞれに対し，12カ月ローリングで自己回帰性の検定（Ljung-Box検定）のP値，
12ヵ月ローリングでのARモデル当てはめの次数（AICによる次数選択）を行った結果
Upper figure : P-value for testing Hypothesis “No Correlation” .　Lower :Order of AR by AIC

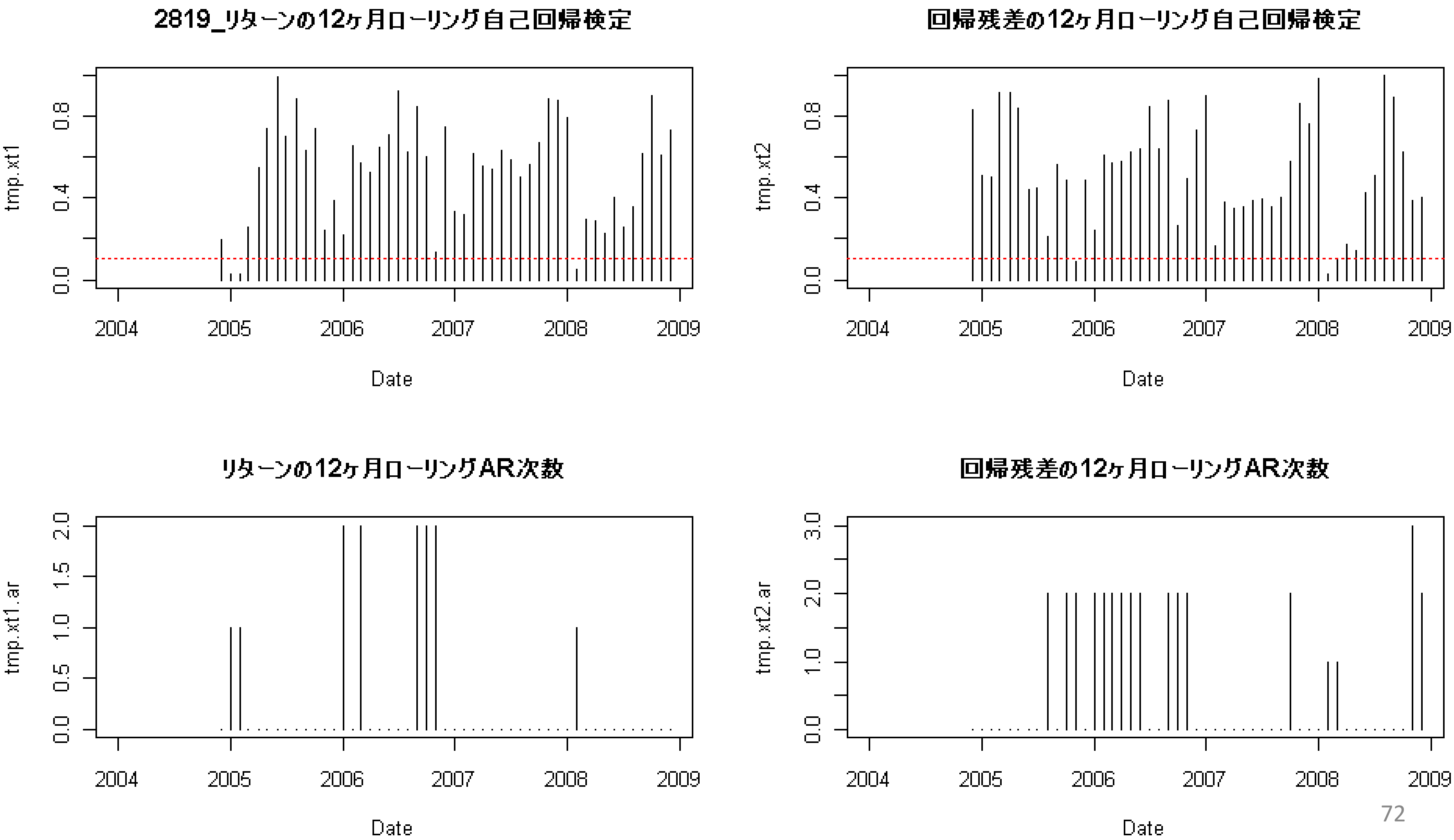


AR in residuals in the last regression model. 7144

Upper figure : P-value for testing Hypothesis “No Correlation” .　Lower :Order of AR by AIC

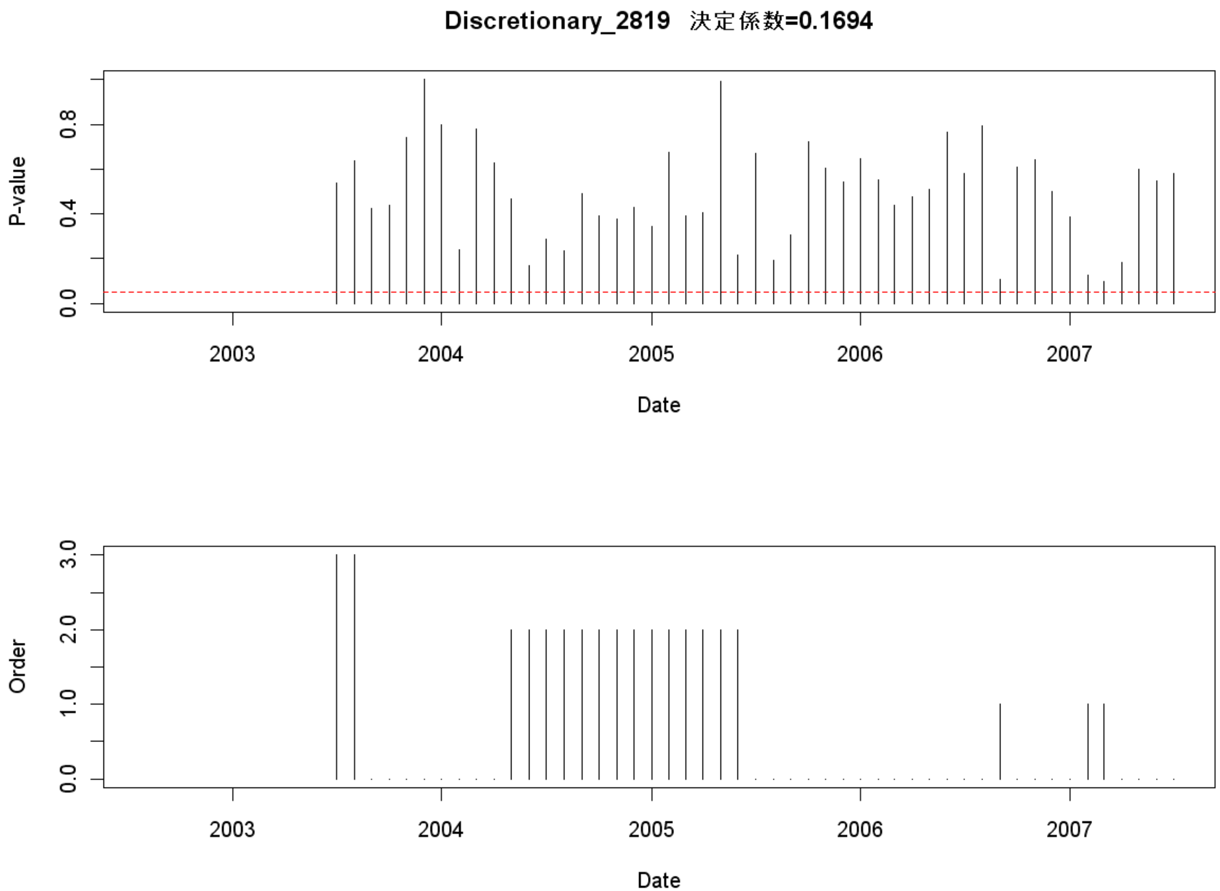


ファンドのリターン系列，「株，債券，為替のﾌｧｸﾀﾓﾃﾞﾙに回帰した時の残差」，
それぞれに対し，12カ月ローリングで自己回帰性の検定（Ljung-Box検定）のP値，
12ヵ月ローリングでのARモデル当てはめの次数（AICによる次数選択）を行った結果
Upper figure : P-value for testing Hypothesis “No Correlation” .　Lower :Order of AR by AIC



AR in residuals in the last regression model. 2819

Upper figure : P-value for testing Hypothesis “No Correlation” .   Lower :Order of AR by AIC



Tentative Conclusion for Part 2.

We see that Coefficients of Determinants (adjusted R-square)are improved by adding more group of Factors, though not fully explained.

But,

The residuals still have AR structures.

So, there must be other factors, such as returns of technical trading strategy, might be tried to explain it.

Also, we note that Intercepts in the regression have not been explained by these factors,

since the variability has not fully explained by these market-available factors.

We will look into this Intercept in the following.

Part 2.

Return Regressions and for Intercepts

60 months data (Aug.2002– Jul.2007)

Our View.

In a Regression with Least Squares methods, the estimated intercepts may be roughly interpreted as the difference between Average returns of each Hedge Funds and the factor contributions ; sum of{ (factor average)x(estimated coefficients)}.

So, we look at the estimated intercepts in the four(or three) models.

（Discretionary and Emerging Markets）

Emerging Markets

:

Change in the estimated Intercept model1 through 4(or3).

This change reveals how much

,(estimated Beta)x(average of the factor variable),

the factor takes out of the average return, as shown in the definition of intercept.

In other word,

how much alpha the Hedge Fund takes beyond the market=available factor (regression independent variables) does.

We should look at how much the estimated intercept has decreased

;how much in each step, the factor took.

Test : Hypothesis "Intercept = 0"

If Intercept is zero, it means that alpha is taken by the factor variables, i.e. the hedge fund CAN BE REPLICATED by them in terms of expected value. The error amount can be seen from the residuals in this regression analysis.

The intercepts decreases in each steps.

There are negative intercepts, which means that in Least square method, intercept seems forced to be negative in order to fit best (estimate beta and intercept) at the same time.

We do not need to interpret the estimated value of intercept in these cases because it may outside of scattered data points area.

P-values for Hypothesis: Intercept=0 : Emerging Markets 55 funds

Rt Returns

: # of funds (all P-values are less than 0.05) are 7 * :
# of funds (P-values are rather small) are 7 ==

: # of funds (all P-values are far larger than 0.05) are the rests

:*******************************

μ+AR(+)

All small: less than either 0.05 or 0.08, are 40 Funds .

All are rather small, are 10 funds.

A Characteristic of Hedge Funds
: Emerging Markets

Autoregressive nature of Monthly Returns.

:**************************

Intercepts of Autoregressive part of returns , in the regression, are

Statistically Significant.

But,

Intercepts of raw Returns

Are not so significant.

What does this mean ?

Next, Discretionary

:

P-values for Hypothesis: Intercept=0 :Discretionary 20 funds

Rt Returns

****

: 4 funds have small P-values and

:3 funds have rather small ones.

:***

μ+AR(+)

****

: Almost All 20 funds (except two) have very small P-values

in the regressions where AR part of returns are regressed.

****

****

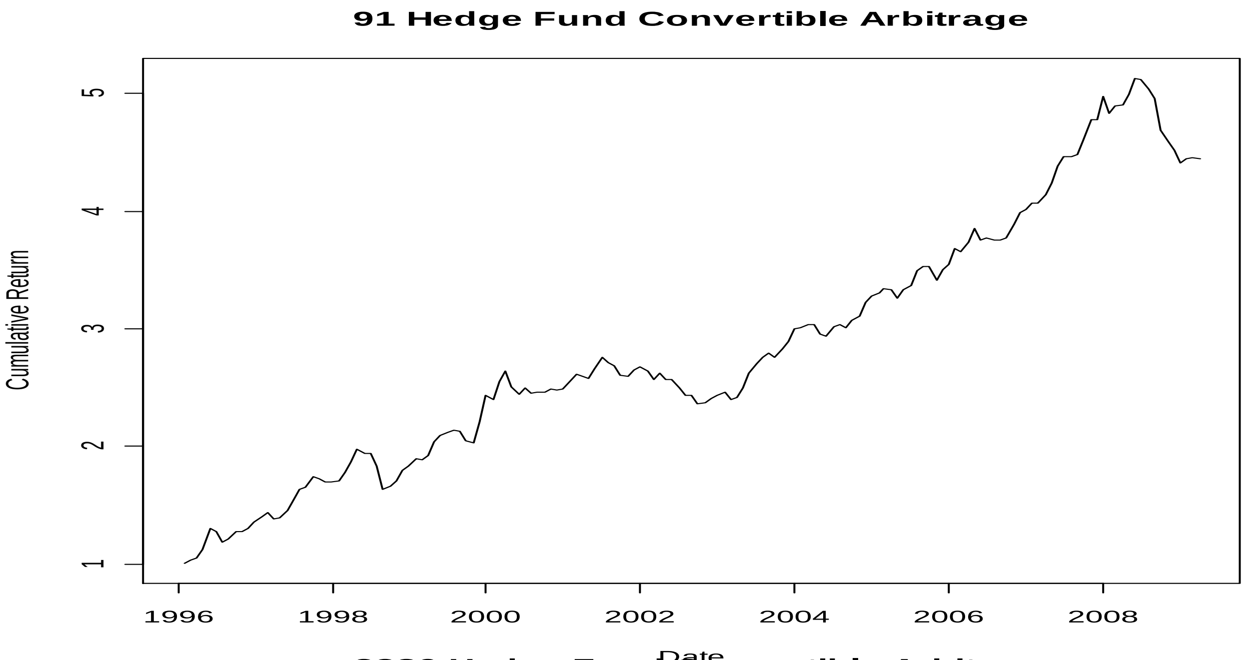
: So, Discretionary is more like Hedge Fund than Emergency Markets ?

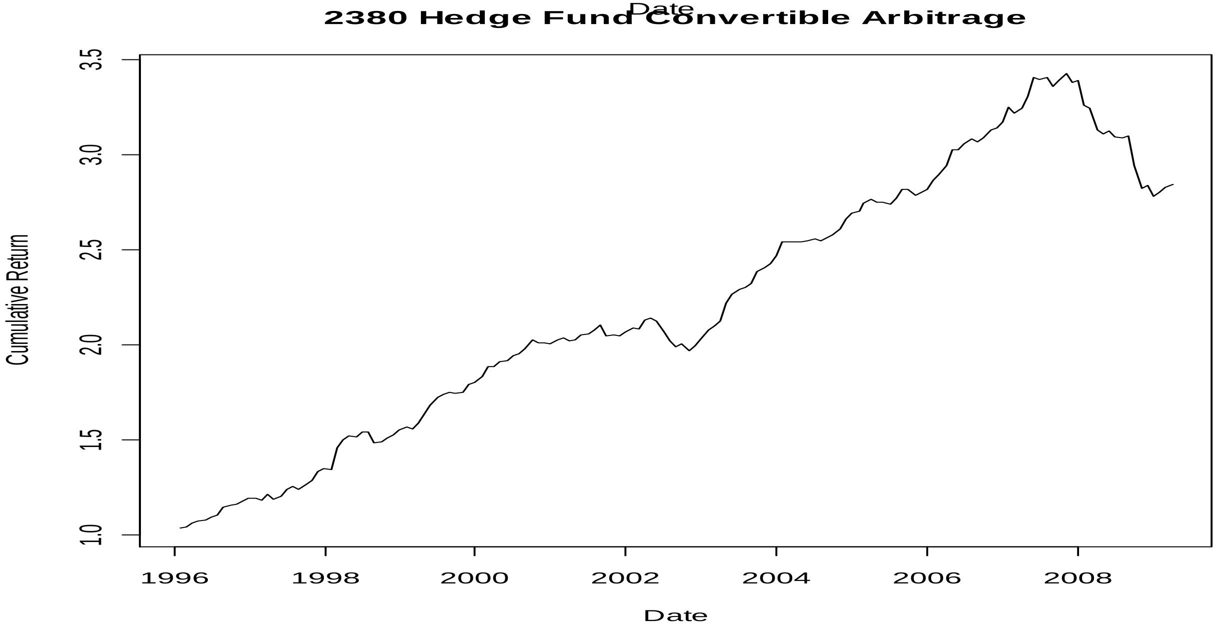
:

- ﾌｧｸﾀ名　ﾌｧｸﾀの６０カ月平均
- **Average of Factor variables**
- **MSCI_WI 0.011131433**
- **EMI 0.023584299**
- **WGBI 0.002689500**
- **ESBI 0.012380917**
- **FX.PC1 -0.015704190**
- **FX.PC2 -0.007962332**
- **FX.PC3 -0.004358038**
- **PTFSBD -0.055600041**
- **PTFSFX 0.004591759**
- **PTFSCOM -0.004099968**
- **PTFSIR -0.038718039**
- **PTFSSTK -0.053475375**

Section 3-2.

: Trials for Data Analysis based on Brownian Nonparametric Statistics.


91 Hedge Fund Convertible Arbitrage
Cumulative Return
1
2
3
4
5
1996
1998
2000
2002
2004
2006
2008
Date


2380 Hedge Fund Convertible Arbitrage
Cumulative Return
1.0
1.5
2.0
2.5
3.0
3.5
1996
1998
2000
2002
2004
2006
2008
Date

Example.

Sample size n=6.

Time-Points: $t_1 < t_2 < t_3 < t_4 < t_5 < t_6$.

Observations: $X_{t1}, X_{t2}, X_{t3}, X_{t4}, X_{t5}, X_{t6}$.

:

Consider a case where

$X_{t3}< X_{t4}<X_{t1}<X_{t5}<X_{t2} <X_{t6}$; start-up-down-up-up-up.

:***

α_i =i/6, $m(\alpha_i)$, Rank(t_i) ,i=1,2,…,6

	$X_{t3}<$	$X_{t4}<$	$X_{t1}<$	$X_{t5}<$	$X_{t2}<$	X_{t6}
$m(\alpha_i)$,	$m(\alpha_1)$,	$m(\alpha_2)$,	$m(\alpha_3)$,	$m(\alpha_4)$,	$m(\alpha_5)$,	$m(\alpha_6)$,
Mapped Time	t_3 ,	t_4 ,	t_1 ,	t_5 ,	t_2 ,	t_6 ,
Rank	1,	2,	3,	4,	5,	6

:***

	t_1,	t_2,	t_3,	t_4,	t_5,	t_6,
Rank(t_i)	3,	5,	1,	2,	4,	6

Figures illustrating the defined times and values

:（2）Two Sample:

[Cumulative Returns of Fund 91]

And[Cumulative Returns of Fund 2380]

(m(α,X)-time, m(α,Y)-time).

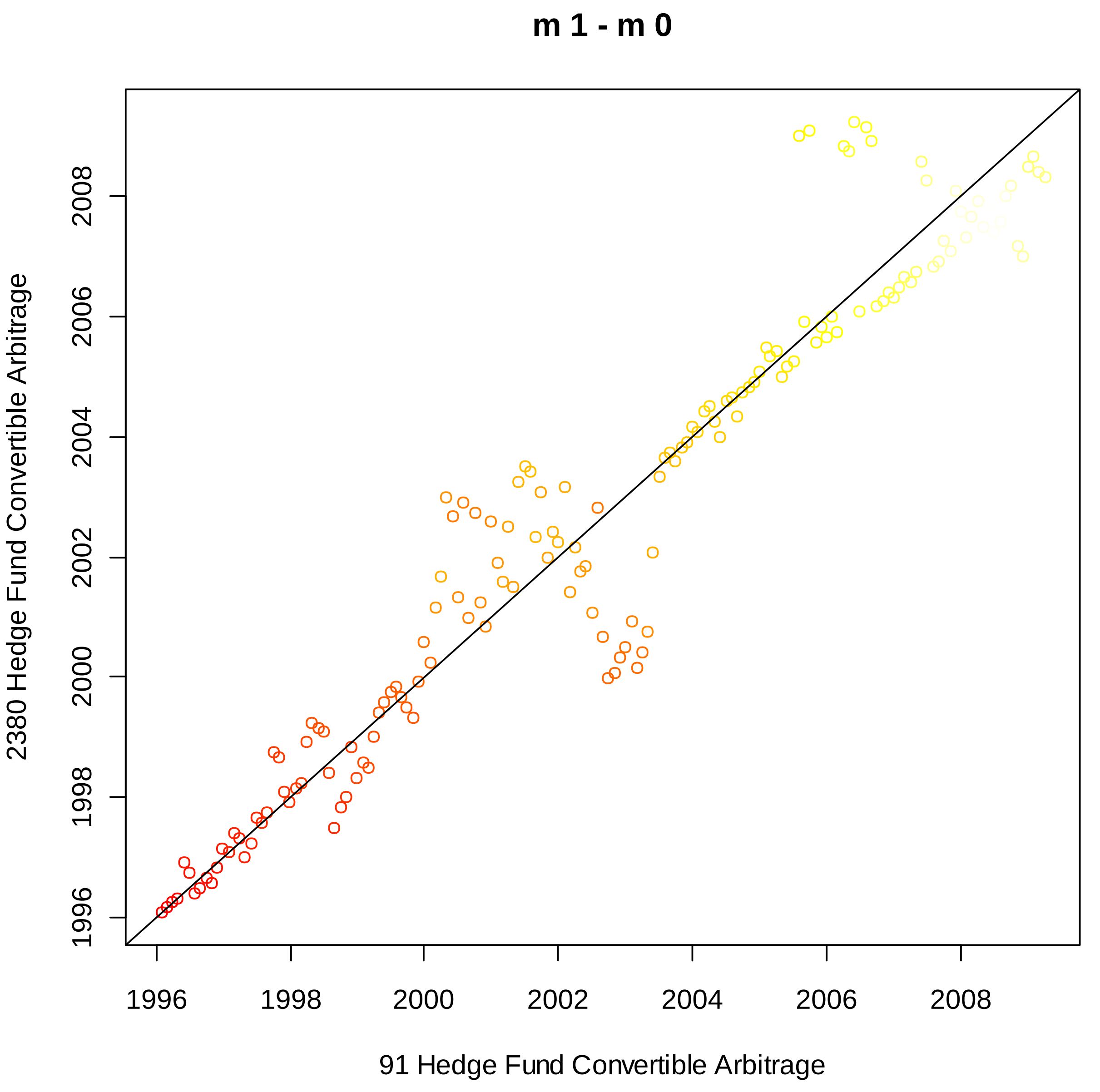
Note: values of α are indicated by colors.

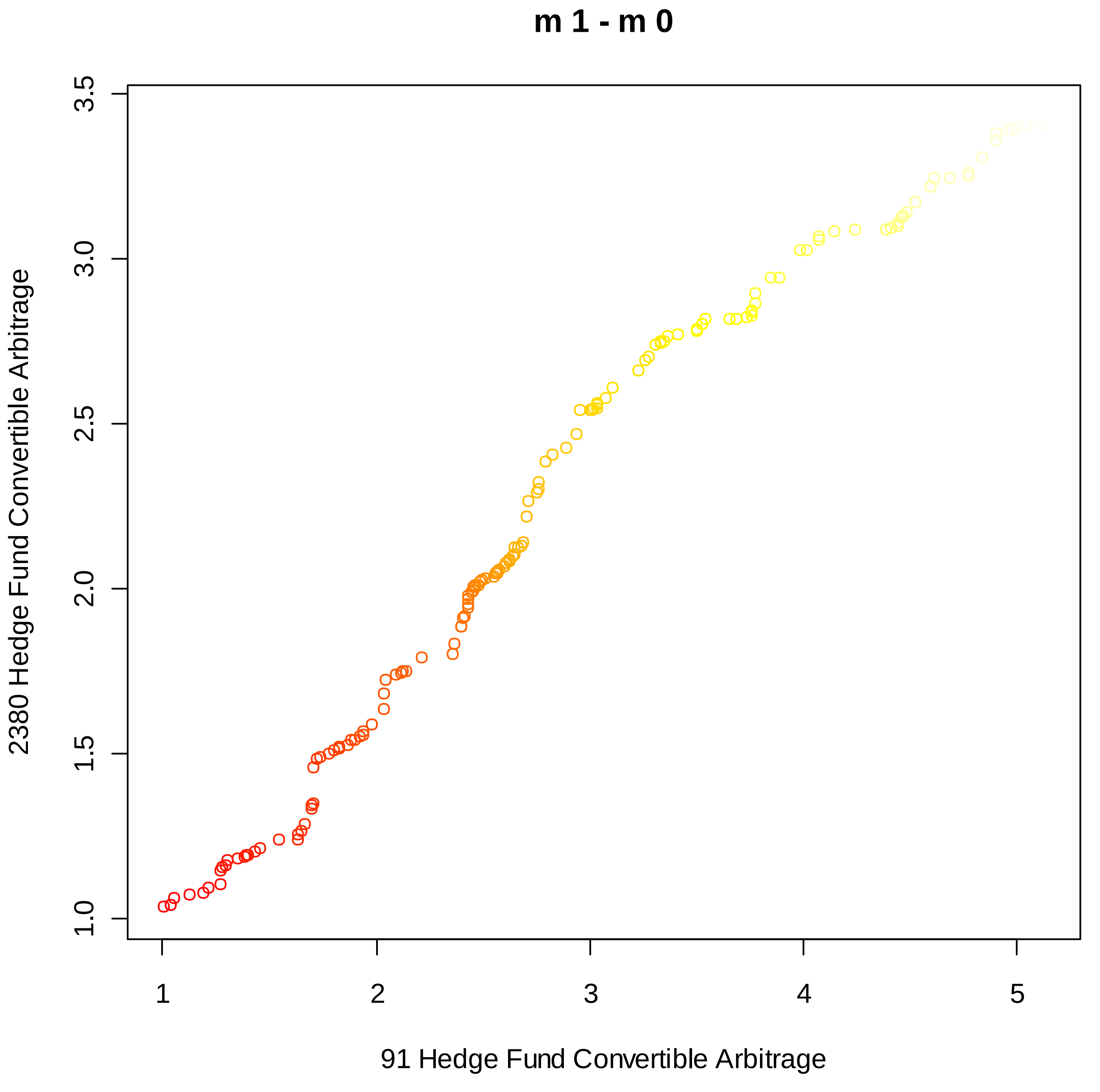
(m(α,X)-value, m(α,Y)-value).

Note: values of α are indicated by colors.

91 and 2380

Convertible Arbitrage.


m 1 - m 0
1996
1998
2000
2002
2004
2006
2008
2380 Hedge Fund Convertible Arbitrage
91 Hedge Fund Convertible Arbitrage


m 1 - m 0
2380 Hedge Fund Convertible Arbitrage
91 Hedge Fund Convertible Arbitrage
1.0
1.5
2.0
2.5
3.0
3.5
1
2
3
4
5

Recall the Statistical Properties. Invariance under Monotone Transformations

:Utilize the good properties of **nonparametric** quantities.

:Use monotone function φ to transform X on Real line [or, equivalently, h(.) to transform F(X) on [0,1] and reverse them back to a real line] .

Note:

:Time-maps are **invariant** under the monotone transformation.

:Ranks do not change.

:Quantiles m(α) changes in their values (**equi-variant**), but **time-map does not change.**

:**************

: Invariance and equi-variance

under location shifts, scale change and monotone transformations.

A Trial for Data Analysis

Some Trials by Principal Component Analysis.

Classifying the Hedge Funds

Principal Component Analysis and Cluster Analysis.

: Using a time series of Ranks of Cumulated returns of a fund for each fund.

: Or use a time series of cumulative returns.

: Ranks seem to be able to distinguish the categories of Strategies!?   IT WORKS !?

:The closeness measured here will be equivalently measured by “ Rank Correlation”. But, we do not know yet a probability distribution of this “ Rank Correlation” statistics.

Note that Ranks use only the information of relative magnitudes of the values. Also, note that Ranks are invariant under monotone transformations, scale change and location shift.

Principal Component Analysisi

Strategy Category:

Convertible Arbitrage &

Emerging Markets

Data Period: January 1999 to December 2004

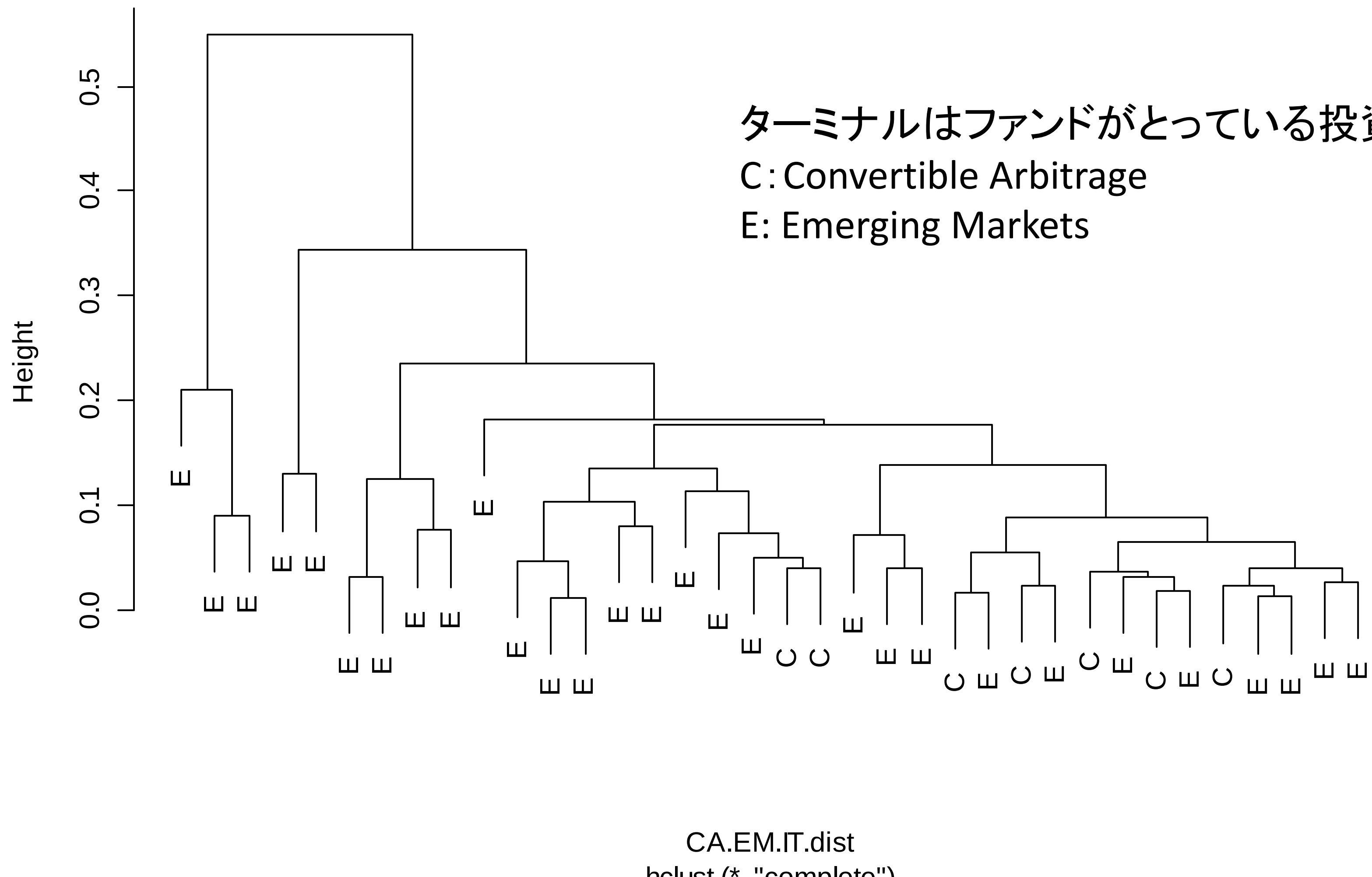
Data: Cumulative Returns

: Ranks

: Mapped Times

Clustering based on Principal Component (EigenValues 5values for each hedge fund) Data: Cumulative Returns

距離は euclid norm
データは第1から第5主成分負荷量までを用いた

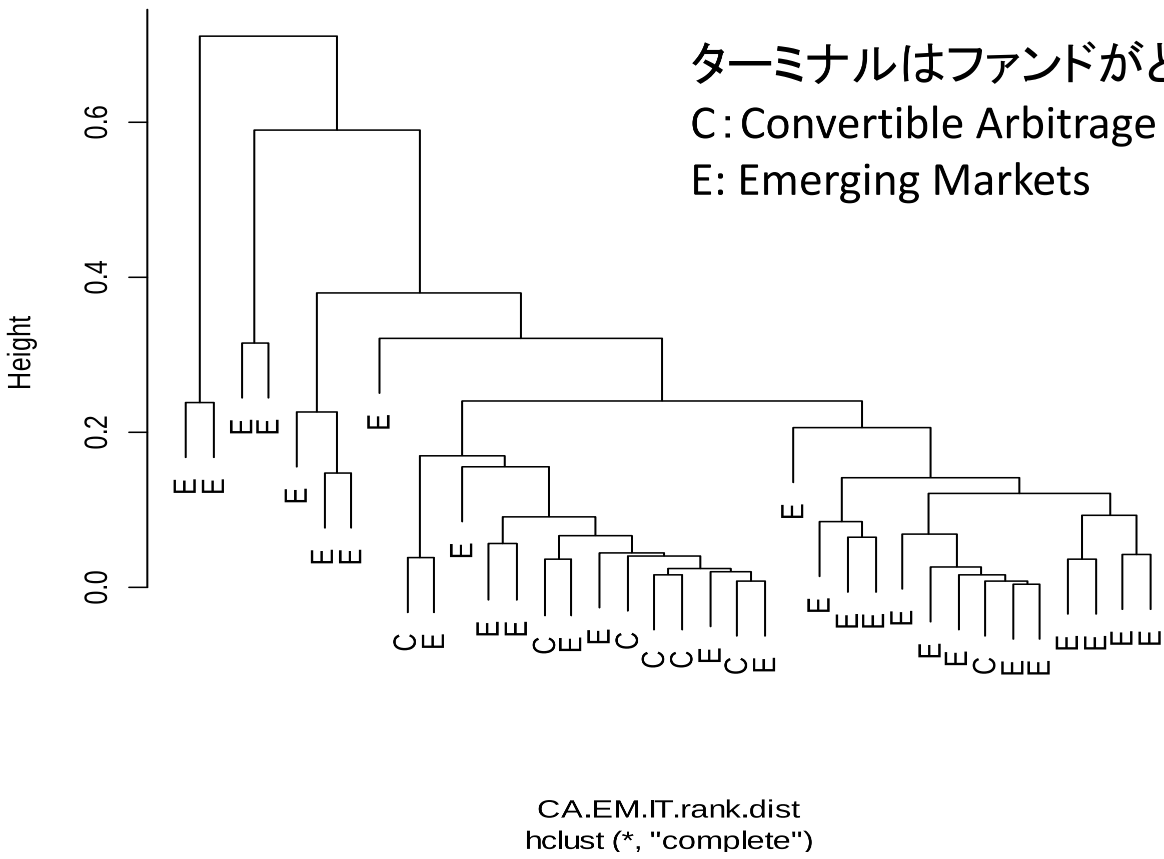


ターミナルはファンドがとっている投資戦略名
C：Convertible Arbitrage
E: Emerging Markets

Clustering based on Principal Components (EigenValues 5values for each Hedge Fund) Data: Rank of Cumulative Returns

距離は euclid norm
データは第1から第5主成分負荷量までを用いた

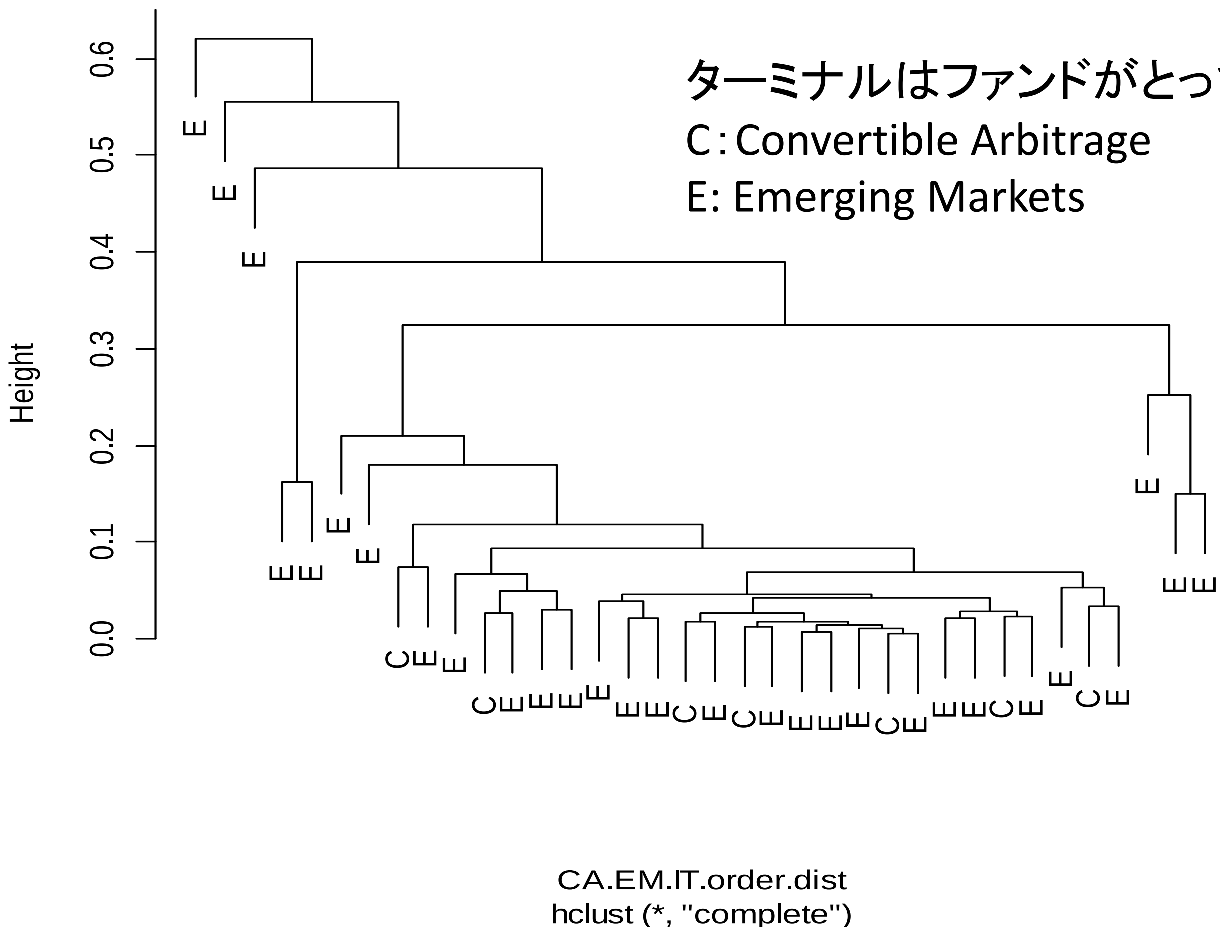
ターミナルはファンドがとっている投資戦略名
C：Convertible Arbitrage
E: Emerging Markets



Clustering based on Principal Components (EigenValues 5values for each Hedge Fund)
Data: Mapped Time from Rank of Cumulative Returns

距離は euclid norm
データは第1から第5主成分負荷量までを用いた

ターミナルはファンドがとっている投資戦略名
C：Convertible Arbitrage
E: Emerging Markets



Section 4. Using Spacings

: Introductory intuitive explanation of the situation where this was used.

: Then, we estimate the asymptotic variance of truncated Hodges-Lehmann's estimator.

Adaptive estimation of location of symmetry

: Huber's Gross error model centered at Logistic distribution.

: Least favorable distribution.

: Truncated Hodges-Lehmann's estimator.

: Adaptive <==> the suitable amount of truncation, which minimizes the asymptotic variance of estimators among the family members.

Miura(1982)

$$\frac{1}{n}\sum a\left(\frac{R_i(\theta)}{n+1}\right)\operatorname{sgn}(X_i-\theta)\quad\left(\triangleq T_n(\theta)\right)$$

$$a(t)=-\frac{f_0'(F_0^{-1}((t+1)/2))}{f_0(F_0^{-1}((t+1)/2))}$$

For the above g and any ε such that $0<\varepsilon<1$, define g_ε, the density function of the Huber's least favorable distribution, as follows.

$$g_\varepsilon(x)=\begin{cases}(1-\varepsilon)\,g(-x_0)\,e^{k(x+x_0)} & \text{for}\quad x\leqq -x_0\\ (1-\varepsilon)\,g(x) & \text{for}\quad -x_0<x\leqq x_0\\ (1-\varepsilon)\,g(x_0)\,e^{-k(x-x_0)} & \text{for}\quad x_0<x\end{cases}$$

where $x_0>0$ and $k>0$. x_0 and k are determined by the relations

(i) x_0 and $-x_0$ are the endpoints of the interval where $|g'/g|\leqq k$

(ii) $\displaystyle\int_{-x_0}^{x_0} g(x)\,dx+\frac{2g(x_0)}{k}=\frac{1}{1-\varepsilon}$.

In correspondence to the above g_ε, define

$$a_{g_\varepsilon}(t) = J_{g_\varepsilon}\left(\frac{t+1}{2}\right) \quad \text{for } 0 \leqq t \leqq 1$$

where

$$J_{g_\varepsilon}(u) = -\frac{g_\varepsilon'(G_\varepsilon^{-1}(u))}{g_\varepsilon(G_\varepsilon^{-1}(u))}$$

$$= \begin{cases} -k & \text{for } 0 \leqq u \leqq \alpha \\ -\dfrac{g'(G_\varepsilon^{-1}(u))}{g(G_\varepsilon^{-1}(u))} & \text{for } \alpha < u \leqq 1-\alpha \\ k & \text{for } 1-\alpha < u \leqq 1 . \end{cases}$$

and $\alpha = \int_{x_0}^{\infty} g_\varepsilon(x)\, dx.$

G_ε is the distribution function of the density g_ε and relates to G by

$$G_\varepsilon^{-1}(t) = G^{-1}\left(\frac{2t-\varepsilon}{2(1-\varepsilon)}\right) \quad \text{for } 0 \leqq t \leqq 1 .$$

We note that $2G(-x_0) = 1 - \dfrac{1-2\alpha}{1-\varepsilon}$.

We write a_ε or a_α instead of a_{g_ε}, and J_ε or J_α instead of J_{g_ε} for the notational simplicity.

Let α_1, α_2 be any numbers such that $0 < \alpha_1 < \alpha_2 < 1/2$. Let $C = \{a_\alpha, \alpha \in [\alpha_1, \alpha_2]\}$.

3.2. Selection rules.

For all $\alpha \in [\alpha_1, \alpha_2]$, the asymptotic length of the confidence interval derived from the signed rank statistic with the score function a_α, multiplied by $\sqrt{n}$, is $t_r \left\{\int_0^1 a_\alpha^2(u)\, du\right\}^{1/2} / \Delta(a_\alpha)$ by Theorem 1. Our interest is to choose α which minimizes $\sigma(\alpha) \equiv \left\{\int_0^1 a_\alpha^2\, du\right\}^{1/2} / \Delta(\alpha)$, where $\Delta(\alpha) \equiv \Delta(a_\alpha)$ for simplicity. Let α_0 be such that $\sigma(\alpha_0) = \underset{\alpha \in [\alpha_1, \alpha_2]}{\text{minimum}}\ \sigma(\alpha)$. Note that such α_0 exists by the continuity of $\sigma(\alpha)$ on the compact set $[\alpha_1, \alpha_2]$. Since the numerator is known, $\Delta(\alpha)$ is the quantity to be estimated. The unknown factor in $\Delta(\alpha)$ is F_0.

3.3. An Example.

For the sake of applications of the adaptive procedure, we illustrate the simplest (trimmed Wilcoxon) case and give some comments.

When G is logistic, i.e. $G(x)=(1+e^{-x})^{-1}$ for $-\infty<x<\infty$, $J(t)=2t-1$ and $a(t)=t$ for $0\leqq t\leqq 1$.

For $0<\varepsilon<1$, we have by definition,

$$J_{\varepsilon}(t)=\begin{cases}(2\alpha-1)/(1-\varepsilon) & \text{for } 0\leqq t\leqq\alpha\\ (2t-1)/(1-\varepsilon) & \text{for } \alpha<t\leqq 1-\alpha\\ (1-2\alpha)/(1-\varepsilon) & \text{for } 1-\alpha<t\leqq 1\end{cases}$$

and

$$a_\varepsilon(t) = \begin{cases} t/(1-\varepsilon) & \text{for } 0 \leqq t \leqq 1-2\alpha \\ (1-2\alpha)/(1-\varepsilon) & \text{for } 1-2\alpha < t \leqq 1. \end{cases}$$

The statistics with these scores are called "trimmed Wilcoxon".
The relation of the constants α, ε, x_0 and k is given by the equations

$$\varepsilon = (1-k)^2/(1+k^2), \quad x_0 = \log((1+k)/(1-k))$$

and $k = \sqrt{(1-2\alpha)/(1+2\alpha)}$.

It is easily seen that when any one of the constants is given, it determines all the others uniquely. The above relation implies $1-\varepsilon = \sqrt{1-4\alpha^2}$. Therefore the trimmed score function for the present case is for $0<\alpha<1/2$,

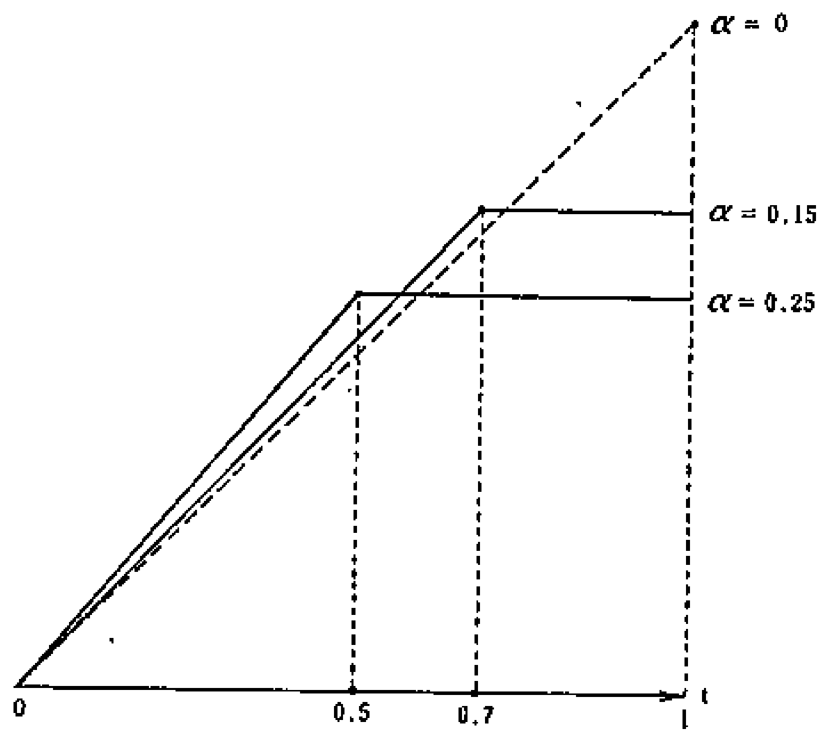
$$a_\alpha(t) = \begin{cases} t/\sqrt{1-4\alpha^2} & \text{for } 0 \leqq t \leqq 1-2\alpha \\ \sqrt{(1-2\alpha)/(1+2\alpha)} & \text{for } 1-2\alpha < t \leqq 1. \end{cases}$$

α indicates the amount of trimming.

The graph of a_α is given in the Figure 2.

Figure 3 displays the probability distributions of $T_n(\theta_0)\times n$ in the two cases where the score functions are the above defined a_α with $\alpha=0$ and $\alpha=$

Figure 2. Graph of $a_\alpha(t)$.



Scand J Statist 8: 48–54, 1981

Spacing Estimation of the Asymptotic Variance of Trimmed Rank Estimators of Location

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Osaka City University

Received January 1978, in final form September 1980

1. Introduction

Let $X_1, \ldots, X_n$ be i.i.d. real random variables with unknown density f, symmetric about the location parameter ξ. Let $\hat{\xi}_J$ be the rank estimator of ξ based on the score function J, introduced by Hodges & Lehmann (1963).

The estimation problem of the asymptotic variance

$$\sigma^2(\hat{\xi}_J) = \left\{ \int_0^1 j(t) f(F^{-1}(t))\, dt \right\}^{-2} \int_0^1 J(t)^2\, dt$$

of $\hat{\xi}_J$, where $j = J'$, has been studied by several authors. See Schweder (1975) for a view on this problem. In this paper, we study the estimation of $\sigma^2(\hat{\xi}_J)$ when J is of a special type.

For a fixed score function J, and for each α such that $0 < \alpha < 1/2$, define the α-trimmed score function J_α of J by

$$J_\alpha(t) = \begin{cases} K & \text{for} \quad 1-\alpha < t \leqslant 1 \\ CJ(t) & \text{for} \quad \alpha < t \leqslant 1-\alpha \\ -K & \text{for} \quad 0 \leqslant t \leqslant \alpha \end{cases}$$

where K and C are constants depending on α. J_α corresponds to the Huber's least favorable distribution, introduced and studied by Huber (1964) and Jaeckel (1971*a*). The asymptotic variance $\sigma^2(\hat{\xi}_{J\alpha})$ of $\hat{\xi}_{J\alpha}$ is a quantity of interest in the adaptive estimation procedure studied by Jaeckel (1971*b*) and Miura (1976). There, $\sigma^2(\xi_{J\alpha})$ is estimated for a fixed J and a discrete set of the values of α, so as to give an order of preference to these values of α.

Schweder (1975) studied the estimator of the asymptotic variance, based on window estimates of the underlying density, which works for a wide class of score functions including the trimmed score functions. The spacing estimator we will study is easier to compute than the window estimator. We will show that the two estimators are asymptotically $n^{1/2}$-order equivalent.

Let $X(i)$ be the ith order statistic, then $2\Delta n^{-1}/\{X(i+\Delta)-X(i-\Delta)\}$ is the spacing estimate of $f(F^{-1}(i/n))$, where Δ is an integer-valued function of n. Let

$$\theta_\alpha = \int_\alpha^{1-\alpha} j_\alpha(t) f(F^{-1}(t))\, dt$$

then our estimate $\hat{\theta}_n(\alpha, \Delta)$ of θ_α is

$$\hat{\theta}_n(\alpha, \Delta) = \frac{1}{n} \sum_{i=n_1}^{n_2} j\left(\frac{t}{n}\right) \frac{2\Delta n^{-1}}{X(i+\Delta) - X(i-\Delta)}$$

where $n_1 = [n\alpha] + 1$, $n_2 = n - [n\alpha]$ and $[n\alpha] \geqslant \Delta$. $[x]$ denotes the largest integer not exceeding x.

Lemma 1. *The rank estimator* $\hat{\xi}_{J\alpha}$ *for* $0<\alpha<\frac{1}{2}$ *is uncorrelated with the spacing estimator*

$$\hat{\sigma}(\hat{\xi}_{J\alpha})=\left[\int_0^1 J_\alpha^2(u)\,du\right]^{1/2}\Big/\hat{\theta}_n(\alpha,\bar{\Delta})$$

of the asymptotic standard deviation of $\hat{\xi}_{J\alpha}$, *provided* J_α *satisfies* $J_\alpha(t)=-J_\alpha(1-t)$.

Proof. By the definition of $\hat{\theta}_n$, and by location and scale equivariance of $\hat{\xi}_{J\alpha}$,

$$\hat{\xi}_{J\alpha}(\xi+X_1,\ldots,\xi+X_n)=2\xi-\hat{\xi}_{J\alpha}(\xi-X_1,\ldots,\xi-X_n)$$
$$\hat{\theta}_n(\xi+X_1,\ldots,\xi+X_n)=\hat{\theta}_n(\xi-X_1,\ldots,\xi-X_n).$$

$\hat{\xi}_{J\alpha}$ and $\hat{\theta}_n(\alpha, \Delta)$ are odd and even functions around ξ, respectively. Since the distribution is symmetric around ξ, $\hat{\xi}_{J\alpha}$ and $\hat{\theta}_n(\alpha, \Delta)$, or $\hat{\sigma}(\hat{\xi}_{J\alpha})$ are uncorrelated.

When $\hat{\xi}_{J\alpha}$ and $\hat{\theta}_n(\alpha, \Delta)$ have asymptotic joint normal distribution, they, and hence $\hat{\xi}_{J\alpha}$ and $\hat{\sigma}(\hat{\xi}_{J\alpha})$ are asymptotically independent. The estimate $\hat{\sigma}(\hat{\xi}_{J\alpha})$ may be used to standardize $\hat{\xi}_{J\alpha}$ just as Schweder (1975) suggests for his window estimator.

2. Asymptotic distribution

In order to study the asymptotic behavior of the spacing estimators, we adopt the method of Pyke & Shorack (1968) and Shorack (1972).

Let $(\Omega, \mathcal{B}, P)$ be a probability space with the special random variables $U_1, \ldots, U_n$ defined on it, where the U_i's are independent and uniformly distributed on the interval $[0, 1]$. $U(i)$'s denote the ordered U_i's and Γ_n denotes the empirical distribution function of them. We denote for each $t \in [0, 1]$,

$$W_n(t) = n^{\frac{1}{2}}(\Gamma_n(t) - t) \quad \text{and} \quad V_n(t) = n^{\frac{1}{2}}(\Gamma_n^{-1}(t) - t).$$

It holds that there exist processes W_0 and V_0 on $[0, 1]$ such that $W_0 \equiv -V_0$, $\sup\{|W_n(t) - W_0(t)| : t \in [0, 1]\} \xrightarrow{e} 0$ and $\sup\{|V_n(t) - V_0(t)| : t \in [0, 1]\} \xrightarrow{e} 0$, where $\xrightarrow{e}$ denotes convergence everywhere on Ω. W_0 (and V_0) is a Brownian Bridge on $[0, 1]$, that is, a Gaussian process with mean zero and covariance function $s(1-t)$, $0 \leqslant s \leqslant t \leqslant 1$.

We first derive a uniform bound for the spacings $U(i+\Delta)-U(i-\Delta)$. Let α_1 and α_2 be any given numbers such that $0<\alpha_1<\alpha_2<\frac{1}{2}$. Let $\underline{n}=[n\alpha_1]+1$ and $\bar{n}=n-[n\alpha_1]$. Note that $\underline{n}\leqslant[n\alpha]+1$ and $\bar{n}\geqslant n-[n\alpha]$ for all $\alpha\in[\alpha_1, \alpha_2]$. Let, for $i=[n\alpha]+1, \ldots, n-[n\alpha]$,

$$Y_i=\frac{U(i+\Delta)-U(i-\Delta)}{2\Delta/n}.$$

Lemma 2. *Assume* $\Delta\leqslant[n\alpha_1]$, *then*

$$\begin{aligned}\sup\{|Y_i-1|: i=\underline{n}, \underline{n}+1, \ldots, \bar{n}\}\\ \leqslant O_p(n^{1/4}\Delta^{-1}(\log n)^{1/2}(\log\log n)^{1/4}\\ +O_p\left(\Delta^{-1/2}\left(\log\frac{n}{\Delta}\right)^{1/2}\right).\end{aligned}$$

Proof. Using the identities;

$$U(i) - \frac{i}{n+1} = \Gamma_n^{-1}\left(\frac{i}{n+1}\right) - \frac{i}{n+1}$$
$$= \Gamma_n^{-1}\left(\frac{i}{n+1}\right) - \Gamma_n\left(\Gamma_n^{-1}\left(\frac{i}{n+1}\right)\right)$$
$$- \frac{i}{n(n+1)}$$

we rewrite $Y_i - 1$ as follows.

$$Y_i - 1 = \frac{n^{1/2}}{2\Delta}\left\{V_n\left(\frac{i+\Delta}{n+1}\right) - V_n\left(\frac{i-\Delta}{n+1}\right)\right\} - \frac{1}{n+1}$$
$$= \frac{n^{1/2}}{2\Delta}\left\{-W_n\left(\Gamma_n^{-1}\left(\frac{i+\Delta}{n+1}\right)\right) + W_n\left(\Gamma_n^{-1}\left(\frac{i-\Delta}{n+1}\right)\right)\right\}$$
$$- \frac{2}{n+1},$$

for all $\underline{n} \leqslant i \leqslant \bar{n}$.

Decomposing this difference into five parts, we obtain

$$\begin{aligned}
&\sup_{\underline{n}\leqslant i\leqslant\bar{n}}|Y_i-1|\\
&\leqslant\frac{n^{1/2}}{2\Delta}\left\{\sup_i\left|W_n\left(\Gamma_n^{-1}\left(\frac{i-\Delta}{n+1}\right)\right)-W_0\left(\Gamma_n^{-1}\left(\frac{i-\Delta}{n+1}\right)\right)\right|\right.\\
&\quad+\sup_i\left|W_0\left(\Gamma_n^{-1}\left(\frac{i-\Delta}{n+1}\right)\right)-W_0\left(\frac{i-\Delta}{n+1}\right)\right|\\
&\quad+\sup_i\left|W_0\left(\frac{i-\Delta}{n+1}\right)-W_0\left(\frac{i+\Delta}{n+1}\right)\right|\\
&\quad+\sup_i\left|W_0\left(\frac{i+\Delta}{n+1}\right)-W_0\left(\Gamma_n^{-1}\left(\frac{i+\Delta}{n+1}\right)\right)\right|\\
&\quad\left.+\sup_i\left|W_0\left(\Gamma_n^{-1}\left(\frac{i+\Delta}{n+1}\right)\right)-W_n\left(\Gamma_n^{-1}\left(\frac{i+\Delta}{n+1}\right)\right)\right|\right\}\\
&\quad+\frac{2}{n+1}\\
&=D_{n1}+D_{n2}+D_{n3}+D_{n4}+D_{n5}+\frac{2}{n+1}.
\end{aligned}$$

Since

$$\sup_i \left| \Gamma_n^{-1}\left(\frac{i}{n+1}\right) - \frac{i}{n+1} \right| \leqslant \sup_{0\leqslant t\leqslant 1} |\Gamma_n(t) - t| + \frac{1}{n},$$

it holds that

$$\sup_i \left| \Gamma_n^{-1}\left(\frac{i}{n+1}\right) - \frac{i}{n+1} \right| = O_p(n^{-1/2}).$$

Now an application of Lévy's module (see, e.g. Itō & Mckean (1965), page 36) to D_{n2}, D_{n3} and D_{n4} and that of Brillinger's approximation (see Brillinger (1969) and Bickel & Rosenblatt (1973)) to D_{n1} and D_{n5} imply

$$\begin{aligned}\sup_i |Y_i - 1| \leqslant \frac{n^{1/2}}{2\Delta} \Big\{ & O_p(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \\ & + O_p(n^{-1/4}(\log n)^{1/2}) \\ & + O_p\left(\left(\frac{\Delta}{n}\right)^{1/2}\left(\log\frac{n}{\Delta}\right)^{1/2}\right)\Big\} + \frac{1}{n}.\end{aligned}$$

Hence the result is obtained.

Let

$$\hat{\theta}_n^*(\alpha, \Delta) = \frac{1}{n} \sum_{i=n_1}^{n_2} b\left(\frac{i}{n}\right) \frac{2\Delta n^{-1}}{U(i+\Delta) - U(i-\Delta)}$$

where $b(t) = j(t)f(F^{-1}(t))$. We derive the asymptotic normality of $\hat{\theta}_n^*(\alpha, \Delta)$ preceding the main result.

Lemma 3. *Assume that the second derivative of b exists and is continuous on* $[\alpha, 1-\alpha]$ *for any* α *in* $(0, 1/2)$. *Assume also that* $\Delta \leqslant [n\alpha_1]$ *and* $n^{\frac{1}{2}}\Delta^{-1}\log(n/\Delta) \to 0$ *as* $n \to \infty$. *Then* $n^{\frac{1}{2}}\{\hat{\theta}_n^*(\alpha, \Delta) - \theta_\alpha\}$ *converges in distribution, as* $n \to \infty$, *to a normal random variable with mean zero and variance* σ_1^2 *given by*

$$\begin{aligned}\sigma_1^2 = 2 \iint_{\alpha<s<t<1-\alpha} & b'(t)b'(s)s(1-t)\,ds\,dt \\ & - 2\alpha(b(\alpha) + b(1-\alpha)) \int_\alpha^{1-\alpha} b'(t)\,t\,dt \\ & + 2\alpha b(\alpha) \int_\alpha^{1-\alpha} b'(t)\,dt + \alpha(1-\alpha)\{b(\alpha) + b(1-\alpha)\}^2 \\ & - 2\alpha b(\alpha)\,b(1-\alpha).\end{aligned}$$

Proof. Denote $Y_i = \{U(i+\Delta) - U(i-\Delta)\}/(2\Delta n^{-1})$ as before.

Since $1/Y_i = 2 - Y_i + (Y_i - 1)^2/Y_i$, we have

$$\hat{\theta}_n^*(\alpha, \Delta) = \frac{1}{n} \sum_{i=n_1}^{n_2} b\left(\frac{i}{n}\right) \left\{2 - \frac{U(i+\Delta) - U(i-\Delta)}{2\Delta/n}\right. + l_n$$

$$= \frac{2}{n} \sum_{i=n_1}^{n_2} b\left(\frac{i}{n}\right) - \frac{1}{2\Delta} \sum_{i=n_1}^{n_2} b\left(\frac{i}{n}\right)$$

$$\times \{U(i+\Delta) - U(i-\Delta)\} + l_n$$

$$\rightleftarrows A_n + B_n + l_n$$

where

$$l_n = \frac{1}{n} \sum_{i=n_1}^{n_2} b\left(\frac{i}{n}\right) \frac{(Y_i - 1)^2}{Y_i}.$$

Since b is bounded, $l_n = O_p(\Delta^{-1} (\log (n/\Delta)))$.

Therefore, since $n^{\frac{1}{2}}\Delta^{-1} \log (n/\Delta) \to 0$, $n^{\frac{1}{2}} l_n$ converges to zero in probability as $n \to \infty$. Now

Theorem 1. *Assume that J is twice and F^{-1} is three times continuously differentiable on the open interval* $(0, 1)$. *Assume also that* $n^{\frac{1}{4}}\Delta^{-1}\log(\Delta/n)\to 0$ *and* $\Delta n^{-3/4}\to 0$ *as* $n\to\infty$. *Then* $n^{\frac{1}{2}}\{\hat{\theta}_n(\alpha, \Delta)-\theta_\alpha\}$ *converges in distribution, as* $n\to\infty$, *to a normal random variable with mean zero and variance*

$$\begin{aligned}\sigma^2 = 2\int_\alpha^{1-\alpha}\int_\alpha^t c(t)\,c(s)\,s(1-t)\,ds\,dt \\ - 2\alpha\{b(\alpha)+b(1-\alpha)\}\int_\alpha^{1-\alpha} c(t)\,t\,dt \\ + 2\alpha b(\alpha)\int_\alpha^{1-\alpha} c(t)\,dt + \alpha(1-\alpha)\{b(\alpha)+b(1-\alpha)\}^2 \\ - 2\alpha b(\alpha)\,b(1-\alpha).\end{aligned}$$

where $b(t) = j(t)f(F^{-1}(t))$ *and*

$$c(t) = b'(t) + j(t)\frac{f'(F^{-1}(t))}{f(F^{-1}(t))}.$$

Proof. We identify $X(i)$ with $F^{-1}(U(i))$ for all i's under consideration. Then our estimator is

$$\hat{\theta}_n(\alpha, \Delta) \equiv \frac{1}{n} \sum_{i=n_1}^{n_2} j\left(\frac{i}{n}\right) \frac{2\Delta/n}{F^{-1}(U(i+\Delta)) - F^{-1}(U(i-\Delta))}.$$

Now let

$$C_i = f\left(F^{-1}\left(\frac{i}{n}\right)\right) \frac{F^{-1}(U(i+\Delta)) - F^{-1}(U(i-\Delta))}{U(i+\Delta) - U(i-\Delta)}$$

and let

$$Y_i = \frac{U(i+\Delta) - U(i-\Delta)}{2\Delta/n}$$

as before.

Then, from the equality $1/x = 1 + (1 - x) + (1 - x)^2/x$, we have

$$\hat{\theta}_n(\alpha, \Delta) = \frac{1}{n} \sum_{i=n_1}^{n_2} j\left(\frac{i}{n}\right) f\left(F^{-1}\left(\frac{i}{n}\right)\right) \times \left\{1 + (1 - C_i) + \frac{(1 - C_i)^2}{C_i}\right\} \frac{1}{Y_i}$$

as in Lemma 3.

By Taylor expansion,

$$1 - C_i = -\tfrac{1}{2} g''\left(\frac{i}{n}\right) f\left(F^{-1}\left(\frac{i}{n}\right)\right) \times \left\{U(i+\Delta) - \frac{i+\Delta}{n} + U(i-\Delta) - \frac{i-\Delta}{n}\right\} + l_{ni}$$

where $g(t) \overset{\leftarrow}{=} F^{-1}(t)$ for $t \in (0, 1)$ and

$$l_{ni} = \frac{g'''(a_i^+)\left\{U(i+\Delta) - \dfrac{i}{n}\right\}^3 - g'''(a_i^-)\left\{U(i-\Delta) - \dfrac{i}{n}\right\}^3}{6\{U(i+\Delta) - U(i-\Delta)\}}$$

for a_i^+ between i/n and $U(i+\Delta)$, and a_i^- between $U(i-\Delta)$ and i/n. For Δ such that $\Delta/n^{\frac{1}{2}} \to \infty$ as $n \to \infty$, Lemma 2 implies

$$\sup\{|l_{ni}| : \underline{n} \leqslant i \leqslant \bar{n}\} \leqslant O_p((\Delta/n)^2) \tag{2.2}$$

under our assumption on F^{-1}.

Hence it holds that

$$\sup\{|1-C_i| : \underline{n} \leqslant i \leqslant \bar{n}\} = O_p(n^{-\frac{1}{2}})$$

since $\sup\{n^{\frac{1}{2}}|\Gamma_n^{-1}(t)-t| : \varepsilon < t < 1-\varepsilon\}$ is bounded in probability for any fixed $0 < \varepsilon < \alpha_1$.

Thus

$$\begin{aligned}
n^{1/2}\{\hat{\theta}_n(\alpha, \Delta)-\theta_\alpha\} \\
&= \frac{1}{n^{1/2}} \sum_{i=n_1}^{n_2} \tfrac{1}{2} j\left(\frac{i}{n}\right) \frac{f'(F^{-1}(i/n))}{f(F^{-1}(i/n))} \\
&\quad \times \left\{U(i+\Delta) - \frac{i+\Delta}{n} + U(i-\Delta) - \frac{i-\Delta}{n}\right\} \\
&\quad + n^{1/2}\left\{\frac{1}{n}\sum_{i=n_1}^{n_2} j\left(\frac{i}{n}\right) f\left(F^{-1}\left(\frac{i}{n}\right)\right)\frac{1}{Y_i} - \theta_\alpha\right\} + o_p \\
&\eqqcolon I_n + II_n + o_p.
\end{aligned}$$

II_n converges as $n \to \infty$ everywhere to

II_n converges, as $n \to \infty$, everywhere to

$$\int_{\alpha}^{1-\alpha} b'(t)\, V_0(t)\, dt + b(\alpha)\, V_0(\alpha) - b(1-\alpha)\, V_0(1-\alpha)$$

as shown in the proof of Lemma 3. In the same manner I_n can be shown to converge everywhere to

$$\int_{\alpha}^{1-\alpha} j(t) \frac{f'(F^{-1}(t))}{f(F^{-1}(t))}\, V_0(t)\, dt \quad \text{as } n \to \infty.$$

Therefore, we conclude that $n^{\frac{1}{2}}\{\hat{\theta}_n(\alpha, \Delta) - \theta_\alpha\}$ behaves asymptotically the same as the sum of the above limits:

$$\int_{\alpha}^{1-\alpha} \left\{ b'(t) + j(t) \frac{f'(F^{-1}(t))}{f(F^{-1}(t))} \right\} V_0(t)\, dt + b(\alpha)\, V_0(\alpha) - b(1-\alpha)\, V_0(1-\alpha) \tag{2.3}$$

which is normal with mean zero and the variance σ^2 stated in the theorem.

Remark 1. If $J(t) = -J(1-t)$, then $b(t) = b(1-t)$ and the asymptotic variance σ^2 is given by

$$\sigma^2 = 2\int_\alpha^{1-\alpha} c(t)\,c(s)\,s(1-t)\,ds\,dt$$

$$+ 2\alpha b(\alpha)\int_\alpha^{1-\alpha} c(t)(1-2t)\,dt + 2\alpha(1-2\alpha)\{b(\alpha)\}^2.$$

Remark 4 (Asymptotic equivalence of window- and spacing-estimators). For the type of score j considered in this paper, our spacing estimator is asymptotically $n^{\frac{1}{2}}$-equivalent to the window estimator; let $\hat{\theta}_n(w)$ be the window estimator of θ_α. Schweder (1975, 1981) showed that $n^{\frac{1}{2}}\{\hat{\theta}_n(w)-\theta_\alpha\}$ is asymptotically equivalent to the random variable

$$2\int_0^1 j(t)f(F^{-1}(t))\,W_n(dt)+\int_0^1 f(F^{-1}(t))\,W_n(t)j(dt)$$

where $W_n(t)=n^{\frac{1}{2}}\{F_n(F^{-1}(t))-t\}\equiv n^{\frac{1}{2}}\{\Gamma_n(t)-t\}$.

By integration by parts, this is equal for our j to

$$-2\int_{\alpha}^{1-\alpha} W_n(t)\left\{j'(t)f(F^{-1}(t))+j(t)\frac{f'(F^{-1}(t))}{f(F^{-1}(t))}\right\}dt$$
$$+\int_{\alpha}^{1-\alpha} W_n(t)f(F^{-1}(t))j'(t)\,dt$$
$$+j(1-\alpha)f(F^{-1}(1-\alpha))\,W_n(1-\alpha)$$
$$-j(\alpha)f(F^{-1}(\alpha))\,W_n(\alpha).$$

Under the everywhere convergence of W_n to W_0 $(\equiv -V_0)$, this converges everywhere to (2.3) which is also the everywhere limit of $n^{\frac{1}{2}}\{\hat{\theta}_n(\alpha, \Delta)-\theta_\alpha\}$.

Corollary 1. *For each γ such that $0 \leqslant \gamma < \frac{1}{2}$, let $\Delta = n^{\beta}$ where $\frac{1}{4} + \gamma/2 < \beta < 1 - \gamma/2$. Then, $n^{\gamma}\{\hat{\theta}_n(\alpha, \Delta) - \theta_{\alpha}\}$ converges to zero in probability as $n \to \infty$, uniformly in α such that $\alpha_1 \leqslant \alpha \leqslant \alpha_2$.*

We conclude by pointing out that the bounds in (2.4) and (2.5) are independent of the values α such that $\alpha_1 \leqslant \alpha \leqslant \alpha_2$.

Corollary 1 provides the validity of the use of our spacing estimator in an adaptive procedure proposed by Jaeckel. (See Jaeckel (1971), Lemma 2 and Miura (1976).)

3. Monte Carlo results on the choice of width of spacing

Some Monte Carlo studies have been carried out to see how the performance of the spacing estimator $\hat{\theta}_n(\alpha, \Delta)$ depends on the choice of Δ for each fixed

For each of the samples of estimates, the mean and the square root of the mean squared error were calculated. The empirical results in the case of Cauchy distribution and sample size 40 are presented in Table 1. The results in the other cases all display the same pattern: The mean and the variance are both decreasing in Δ, and they both decreases slowly in the range of $\Delta \geqslant 2$. For each of the four distributions, and for each α cosidered, the value of Δ at which the mean squared error has a minimum was picked up. They are listed in Table 2.

Table 2. *Empirical best choice of* Δ

When $n\alpha=1$, Δ has no other choice than 1

		$n=20$				$n=40$								
Distributions	$n\alpha$ ...	2	3	4	5	2	3	4	5	6	7	8	9	10
Standard normal		2	3	4	5	2	3	4	5	6	7	8	8	9
Logistic		2	3	4	5	2	3	4	5	6	7	7	8	9
Double exponential		2	3	4	4	2	3	4	5	6	6	7	7	7
Cauchy		2	3	4	4	2	3	4	5	5	5	5	6	6

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SPACING ESTIMATION OF THE ASYMPTOTIC VARIANCE OF RANK ESTIMATORS OF LOCATION

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The adaptive trimmed Wilcoxon estimator thus defined has the following property which can be proved in the same way as that of Lemma 1 in Section 1. There we should include that the shift and scale change of observations, $X_i : i = 1, \ldots, n$, and the flipping of them around the center ξ of symmetry do not change the choice $\hat{\alpha}$ of the best trimming proportion.

Lemma 5 : *The adaptive trimmed Wilcoxon estimator* $W_{\hat{\alpha}}$ *for* $0 \leqslant \alpha < \frac{1}{2}$ *is shift and scale equivariant, and is uncorrelated with the spacing estimator* $\hat{\sigma}(W_{\hat{\alpha}}) = \left\{ \int_0^1 J^2_{\hat{\alpha}}(t)dt \right\}^{\frac{1}{2}} \Big/ \hat{\theta}_n(J_{\hat{\alpha}}, K, \Delta)$ *of the asymptotic standard deviation of* $W_{\hat{\alpha}}$*, provided that the second moment of X's exist.*

Comparisons with the adaptive trimmed mean. Let $\bar{X}_\alpha$ denote the trimmed mean and $\bar{X}_{\hat{\alpha}}$ denote the Jaeckel's adaptive trimmed mean.

C1. Define the asymptotic efficiency of W_α with respect to $\bar{X}_\beta$ by the ratio of their asymptotic variances.

$$e_F(W_\alpha, \bar{X}_\beta) = \frac{\sigma_F^2(\bar{X}_\beta)}{\sigma_F^2(W_\alpha)} = \left\{ \int_\beta^{1-\beta} \{F^{-1}(t)\}^2 dt + 2\beta\{F^{-1}(\beta)\}^2 \right\}\left\{ \int_\alpha^{1-\alpha} f(F^{-1}(t))dt \right\}$$

$$\times 12/(1-2\beta)^2(1-2\alpha)^2(1+4\alpha).$$

it is not well known. Its proof is in the spirit of Hodges and Lehmann (1956) and Bickel (1965), and is omitted here.

Let $\mathscr{F}$ be the class of all the symmetric (about zero) distributions satisfying the regularity conditions for the asymptotic normality of the estimates of location. Then,

$$\sup\{e_F(W_\alpha, \bar{X}_\beta) : F \in \mathscr{F}\} = \infty$$

$$\inf\{e_F(W_\alpha, \bar{X}_\beta) : F \in \mathscr{F}\} = \begin{cases} 0{\cdot}864 \times B(\beta)/(1-2\beta)^2(1-2\alpha)^2(1+4\alpha) & \text{for } 0 \leqslant \alpha \leqslant \beta < \tfrac{1}{2} \\ 0{\cdot}864 \times B(\alpha)/(1-2\beta)^2(1-2\alpha)^2(1+4\alpha) & \text{for } 0 \leqslant \beta \leqslant \alpha < \tfrac{1}{2} \end{cases}$$

where

$$B(t) = \left\{1+4t+\frac{4}{3} \times \sqrt{3t(t+1)}\right\}\left\{8t^2+1-\frac{2}{3} \times (1+4t)\sqrt{3t(t+1)}\right\}^2 \quad \text{for } 0 \leqslant t < \tfrac{1}{2}.$$

By a numerical tabulation of the lower bound, we find that for each given β the lower bound is maximized at $\alpha = \beta$ and this maximum value increases strictly to one as $\alpha = \beta$ increases to $\frac{1}{2}$. We find the same thing on the other way around except for small values of α ($< 0{\cdot}04$) where the maximum is attained around $\beta = 0{\cdot}04$.

Bahadur's representation theorem in iid case, and weakly dependent

Spacing estimate was discussed for iid case.

Biao Wu seems to have obtained Bahadur's representation in an extended case (non iid).

Also, there is a new result for a convergence of weakly dependent sequence of random variables.

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Weak convergence for empirical processes of associated sequences

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1. INTRODUCTION, NOTATIONS AND PREVIOUS RESULTS

Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary sequence of random variables (r.v's) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let F be the common distribution function of $(X_n)_{n\in\mathbb{Z}}$. The empirical distribution function F_n of $X_1, \dots, X_n$ is defined as:

$$F_n(x) := F_n(x, \omega) = \frac{1}{n} \sum_{1\leqslant i\leqslant n} \mathbf{1}_{X_i(\omega)\leqslant x}, \quad x \in \mathbb{R}.$$

The empirical process G_n based on the observations $X_1, \dots, X_n$ is defined by:

$$G_n(x) := G_n(x, \omega) = \sqrt{n}\big[F_n(x, \omega) - F(x)\big]. \tag{1}$$

Let $D[-\infty, +\infty]$ be the space of *cadlag* functions on $[-\infty, +\infty]$ having finite limits at $\pm\infty$. Suppose that $D[-\infty, +\infty]$ is equipped with the Skorohod topology. The usual Empirical Central Limit Theorem (ECLT) gives conditions under which the empirical process $\{G_n(x), \ x \in \mathbb{R}\}$ converges in distribution, as a random element of $D[-\infty, +\infty]$, to a Gaussian process G with zero mean and covariance

$$\mathrm{Cov}\big(G(x), G(y)\big) = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(\mathbf{1}_{X_0\leqslant x}, \mathbf{1}_{X_k\leqslant y}). \tag{2}$$

The proof of such theorem requires two steps:

Step 1. Establish the convergence of finite-dimensional distributions.

Step 2. Establish the tightness property.

In general, it remains to prove step 2 since step 1 follows from a suitable central limit theorem, usually well known.

For the sake of simplicity, we suppose in the sequel that the marginal distribution function F is *continuous* on $\mathbb{R}$. This restriction allows to suppose that the marginal law is $\mathcal{U}([0, 1])$: the uniform law over $[0, 1]$ (cf. Billingsley [2]).

Definition.

$\{X_n$, n= 1,2,....$\}$ is a sequence of associate random variables if for every finite subcollection X_{i1}, ..., X_{in} and for every pair of coordinatewise non decreasing functions h(.) and k(.) : $\mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Cov}(h(X_{i1}, ..., X_{in}), k(X_{i1}, ..., X_{in})) \geq 0,$$

whenever the covariance is defined.

Memo: This paper mentions that Gaussian processes are associated if and only if their covariance function is positive, referring a paper(Pitt(1982)).

2. MAIN RESULT AND APPLICATION

THEOREM 1. – *Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary associated sequence with continuous marginal distribution F. Assume that, for $n \in \mathbb{N}^*$,*

$$\mathrm{Cov}\big(F(X_1), F(X_n)\big) = \mathcal{O}\big(n^{-b}\big), \quad \textit{for } b > 4. \tag{10}$$

Then

$$G_n(.) \to G(.) \quad \textit{in } D[-\infty, +\infty],$$

where $G_n(.)$ is defined by (1) *and G is the zero-mean Gaussian process with covariance defined by* (2).

$\mathrm{Cov}(F(X_1), F(X_n))$ can be written as,

$$\mathrm{Cov}(F(X_1), F(X_n)) = \iint f(u)f(s)\mathrm{Cov}(1_{\{X_1 \le u\}}, 1_{\{X_n \le s\}})duds$$

$$\le \| f \|_{\infty}^{2} \mathrm{Cov}(X_1, X_n).$$

$(Memo : (Y1, Yn)$ be independent of (X1,Xn) and be distributed as the same as (X1,Xn).

Then, we have

$$\mathrm{Cov(F(X1,Xn)}=\frac{1}{2}\mathrm{E[\{F(X1)\text{-}F(Y1)\}\{F(Xn)\text{-}F(Yn)\}]}$$

Note: $F(X)=P\{V \le X\}=E_F(1\{V \le X\}) = \int 1\{v \le X\}f(v)dv$

, where V follows the distribution F. Also note;

$$F(x)\text{-}F(y)=\int [1\{v \le x\}\text{-}1\{v \le y\}f(v)dv.$$