3rd Day. May 4, 2011 (3+2 =5hours) [Financial Engineering]

: (i) Some more applications of Ranks and quantiles.

: (ii) Executive Option-like Derivatives.

: (iii) Value at Risk

: (iv) Commodity-Linked Bond.

BDS Statistics.

Linear Stochastic Differential Equations.

References.

R.Miura (5)
"Statistical Methodologies for the Market Risk Measurement" (with Shingo Oue),
Asia-Pacific Financial Markets, No.7, pp305-319, 2000

R.Miura (6).
"The Pricing Formula for Commodity Linked Bonds with Stochastic Convenience Yields and Default Risk" (with Hiroaki Yamauchi),
Asia-Pacific Financial Markets, Vol.5, No.2, pp129-158, 1998

Ishii&Fujita (2010)
“Valuation of Repricable Stock Options.”
Asia-Pacific Financial Markets. 17. 1-18.

Section 1. More on Ranks and Quantiles
: 1-1. Some more applications of Ranks and quantiles
: 1-2. Executive Stock Options

In 1-2, I will give some comments on a recent work.

Ishii and Fujita
(2010 APFM, published online 2009)

“Valuation of a Repriceable Executive Stock Option”

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: My comment is that it is possible to use Ranks to determine a payoff.

There, we can utilize a convenient probabilistic property of Ranks.

: 1-1. Some more applications of Ranks and quantiles

Brownian quantiles

: represent Level of path relating to an occupation time.

Rank

: is not a level of a path, but an occupation time.

Invariance property of Rank.

: Probability distribution of Rank does not depend on the (level of) initial value S_0 .

Designing Exotics

: (1). Determining a knock-out condition.

[0,T] . S starts from time 0.

A bar at the level A. A>0.

τ = the first time S_t hits the level A from below.

We should like to appreciate an early hittings (arrivals).

Then, let $\alpha = \tau / T$, and look at M=m(1- α ,[τ ,T]) to define a conditional exotic pay-off as ; " Win if M>A ".

Note that if α is small, then 1-α is large, and M>A means S can relax.

({win a predetermined pay-off }I{M>=A}
+{lose}(I{M<A}) I{ τ <T}+{lose}I{ τ >T})

This will make the case of early hitting easier to win.

We can set a time deadline T_0, $0< T_0 < T$, or τ, and look at M on [τ ,T]. Make it that one does not obtain the right to receive a pay-off if S does not hit the level A by the time-deadline.

What can be a Pay-off ? Well, any.

Also one can use ranks R(S_T :[τ ,T]) to determine the pay-off.

Note：the probability distribution of m(1- α ,[τ ,T]) with $\alpha = \tau$ /T will be calculated by applying the above explained mathematical results/procedure.

$$P\{m(1-\alpha;[\tau,T]) < x\} =$$

$$= \int_0^T P\{m(1-\alpha;[\tau,T]) < x; \Big| given\ \tau = a, \alpha = \frac{a}{T}\} f_\tau(a)da$$

where f_τ is the density function of τ. Note that α is stochastic.

: (2). Corridors. (Another Stochastic Corridor)

$0 < T_0 < T_1$. Contract is made at time 0.

Fix α at time 0.

Use $m(\alpha, [0, T_0])$ for the level of corridor on $[T_0, T_1]$.

And look at

$$\frac{1}{T_1 - T_0} \int_{T_0}^{T_1} I\{S_u < m(\alpha; [0, T_0])\} du$$

This is another stochastic corridor since the level of corridor is stochastic at time 0 (this remains stochastic until time T_0).

This is actually comparing $m(\alpha, [0, T_0])$ with $m(\alpha, [T_0, T_1])$.

Or, looking at the difference of the two.

The probability distribution can be derived, in a similar way, by taking the conditional distribution function of,

$$\frac{1}{T_1 - T_0}\int_{T_0}^{T_1} I\{S_u < m\}du$$

,
given the value m of m(α ,[0, T_0]),
then, by integrating it with the density of m(α ,[0, T_0]) over [0,∞].

Note: $\{m(\alpha:[T_0,T_1]) < x\} = \{\frac{1}{T_1 - T_0}\int_{T_0}^{T_1} I\{S_u < x\}du > \alpha\}$

Represent the random variable or the events as follows;

$$\frac{1}{T_1 - T_0}\int_{T_0}^{T_1} I\{S_u < x\}du > \alpha <=> \frac{1}{T_1 - T_0}\int_{T_0}^{T_1} I\{S_{T_0}\exp(W_u - W_{T_0}) < x\}du > \alpha$$

$$<=> \frac{1}{T_1 - T_0}\int_{T_0}^{T_1} I\{(W_u - W_{T_0}) < \log(x / S_{T_0}\}du > \alpha$$

Also for $[T_1, T_2]$, where $0<T_0<T_1<T_2$,

Represent the random variable or the events as follows;

$$\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} I\{S_u < y\}du > \beta <=> \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} I\{S_{T_1}\exp(W_u - W_{T_1}) < y\}du > \beta$$

$$<=> \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} I\{(W_u - W_{T_1}) < \log(y / S_{T_1}\}du > \beta$$

Note: $S_{T_0} = S_0 \exp(W_{T_0})$

and $S_{T_1} = S_{T_0}\exp(W_{T_1} - W_{T_0}) = S_0 \exp(W_{T_0})\exp(W_{T_1} - W_{T_0})$

Let $W^1{}_u = W_u - W_{T_0}$, for $T_0 < u < T_1$ and $W^2{}_s = W_s - W_{T_1}$, for $T_1 < s < T_2$.

: Note that $W^1{}_u$ and $W^2{}_s$ are independent.

Then, conditionally (given $W_{T_0} = a, W_{T_1} = b$), the probabilities for

$$\frac{1}{T_1 - T_0}\int_{T_0}^{T_1} I\{(W_u - W_{T_0}) < \log(x / S_{T_0}\}du > \alpha$$

$$<=> \frac{1}{T_1 - T_0}\int_{0}^{T_1 - T_0} I\{W^1{}_u < \log(x / S_0 e^a\}du > \alpha$$

and similarly $\frac{1}{T_2 - T_1}\int_{0}^{T_2 - T_1} I\{W^2{}_s < \log(y / S_0 e^a e^b\}ds > \beta$

can be obtained by reducing these to (set v=u-T_0 for W^1 and v=u-T_1 for W^2)

$\int_0^1 I\{W^1{}_v < \log(x / S_0 e^a\}dv > \alpha$ and $\int_0^1 I\{W^2{}_v < \log(y / S_0 e^a e^b\}dv > \beta$

For $0 < T_0 < T_1 < T_2 < ... < T_n = T$,

a chain of conditional distribution/calculation will come.

: (4). α .

Use the magnitude of α and Rank to judge the percentile level of S during a concerned (first) time interval.

$0 < T_0 < T_1$.
Contract is made at time 0.
Fix α at time 0.
Compare $R(S_{T0}, [0, T_0])$ with α ,
And then let pay-off be determined via $m(\alpha^*, [T_0, T_1])$,
where $\alpha^* = \max\{\alpha, R(S_{T0}, [0, T_0])\}$.

Note that $R(S_{T0}, [0, T_0]) > \alpha$ means $S_{T0} > m(\alpha, [0, T_0])$.

This may be issued for investors who expect the stock price S to go up in the second time interval $[T_0, T_1]$ as "$R(S_{T0}, [0, T_0]) > \alpha$" may indicate so (?).

Homework

Describe an outline of calculation for the price of the security in the above page.

You do not have to calculate everything, but you can end in an integral form as a final form, such as the ones shown in the slides (1st. & 2nd week of this lecture). I think that is what we can do at most for now.

Required to derive Probability Distributions of quantities and/or events ;... **in order to utilize Occupationtime F(K), Brownian quantiles and ranks**

: [1] m(α) minus m(β).

A difference of two quantiles; m(α) and m(β). (Fujita(2000))

: [2] Definitions for defaults, and use it for setting a condition to win (or lose) a right for a pay-off of Knock-In and Knock-Out-type derivatives.

: Fujita, T. and Miura, R.(2002). : Fujita, T. and Ishizaka, M.(2002).

[3] Joint Distributions of two α-quantiles for two time intervals. Also that of two Ranks for two time intervals.

:Miura,R., Fujita,T. and Kamimura,S. (2005) . Presented at QMF05.

They have given the joint probability distribution of two α-quantiles over the two overlapping intervals $[T_1,T_2]$ and $[T_0,T_2](= [T_0,T_1]+ [T_1,T_2])$

We need more probability distribution theory to develop along with this line.

1-2. Applications to Exotic Stock Option

Following Ishii and Fujita(2010),

I propose another design of Stock Option in order to appreciate an effort of managers of the company.

Stock Options and alikes

:Stock options counts the amount of payoffs in terms of a number of shares(stocks), whereas "alikes: stock option-like" measures in other terms such as days with multiplying by a constant (in terms of money units).

Framework of Typical Stock Options.

: **Underlying variables.** (stock price of the firm)

: **Time intervals; $[0,T_0]$ and $[T_0, T_1]$.** (Option be issued at time 0.)

: **Condition for stock price level**

(stock price of the firm should **hit a level L** at least once during $[0,T_0]$.)

(We modify this condition to utilize our nonparametric statistics.)

:**Exercise type.** (if the condition is satisfied, an option holder can exercise any time during $[T_0, T_1]$.)

PAYOFF is $I\{ \max(X_u : u \in [0,T_0]) > L\} \max\{ St-K,0\}$,where t is the time of exercise and K is the exercise price. (This is an American-type option . But today we talk on an European type which can be exercised only at T_1 for simplicity.)

We also replace T_0 by a stopping time τ for some cases.

Using a stopping time τ to start counting for Payoffs.

[0, T] and $0 \le \tau \le T$

We can work on only one underlying variables , by setting

Stock price condition be : For an occupation time of the stock price to stay above a certain level L should be long enough, say , once it hits a certain length M then it stopps (at time τ) and a new counting starts. $[0, \tau]$ and $[\tau, T]$.

Then, **payoff** can be given in a way where one can use a **Rank (S_τ [τ,T])**.

Payoff at an exercise can be **{1- R(S_τ :[τ,T])}xConstant** , in order to appreciate the management for keeping the stock price high, which may imply enough dividend being given to stock share holders.

PAYOFF function is {1- R(S_τ :[τ,T])}xConstant where **$\tau=\inf\{t: F(L: [0,t]) > M/T\}$.**

Now, we may work on two underlying variables such as the case in Ishii and Fujita (2010) where the second stochastic process is a Market index which represents a level of economy (economic activities).

Ishii and Fujita
(2010 APFM, published online 2009)
“Valuation of a Repriceable Executive Stock Option”

Their idea is to appreciate an effort of management during a period of economic recession.

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space, and that $W = (W_1, W_2)$ is a 2-dimensional Brownian motion process. $\{\mathcal{F}_t\}_{t \geq 0}$ denotes the standard Brownian filtration.

Define two stochastic processes X_1 and X_2 as follows: for $\forall t \geq 0$,

$$X_1(t) = x_1 e^{(\mu_1 - \frac{1}{2}\sigma_1{}^2)t + \sigma_1 W_1(t)}, \tag{1}$$

$$X_2(t) = x_2 e^{(\mu_2 - \frac{1}{2}\sigma_2{}^2)t + \sigma_2(\rho W_1(t) + \sqrt{1-\rho^2} W_2(t))}, \tag{2}$$

where $x_1, x_2, \mu_1, \mu_2, \sigma_1$, and σ_2 are positive constants, and $|\rho| < 1$. For any $t \geq 0$, we use $X_1(t)$ as a stock market index (e.g. S & P 500, Dow Jones Averages, FT-SE 100, TOPIX, or NK225) at time t, and $X_2(t)$ as a firm's stock price at time t.

We now present a repriceable executive stock option. Fix $M_1 \in (0, x_1]$ and $M_2 \in (0, x_2]$, and let τ be a stopping time defined by

$$\tau = \inf\{t \geq 0 | X_1(t) \leq M_1,\ X_2(t) \leq M_2\}. \tag{3}$$

Fix $T > 0$, and let us define an $\mathcal{F}_T$-measurable random variable

$$Y = \max(X_2(T) - K, 0)1_{\{\tau > T\}} + \max(X_2(T) - X_2(\tau), 0)1_{\{\tau \leq T\}}, \qquad (4)$$

where K is a positive constant, and 1_A is an indicator function of $A \in \mathcal{F}$. The random variable Y represents the payoff of the executive stock option. We call T the expiration day. K is referred to as an exercise price which is determined at time 0, i.e. the time of stock option issue. We use $X_2(\tau)$ as the new exercise price. The new exercise price is reset to the market price of the firm's share at the repricing date. So, the exercise price may be marked down only one time, if the firm's stock price falls below the boundary M_2 in a down market before the expiration day T.

Comments 1 : Using Brownian quantiles and Ranks to make a device on Design of stopping time.

Ishii and Fujita use hitting times of X_1 and X_2 to define a stopping time. It is fine.

Here, we can define a stopping time from other viewpoints with Ranks and Brownian quantiles.

: [1] First t such that $X_1(t) < M_1$ be τ.

Then, use {Rank of $X_2(\tau)$ in time interval [τ, T]}) and give some care of $\max\{(X_2(\tau)- K),0\}$.

Note that a probability distribution of rank does not depend on the level of $X_2(\tau)$, and probability distribution of $(X_2(\tau)- K)$ can be obtained from joint distribution of $(X_1(\tau), X_2(\tau))$ under the condition of $X_1(\tau)=M_1$ and probability distribution of τ which is a well known hitting time of a Brownian motion.

PAYOFF function is $\{1- R(X_2(\tau):[\tau,T])\} \times \text{Constant}$ where $\tau=\inf\{t: X_2(t)> M\}$.

Some care may be possible, for example, add $\max\{(X_2(\tau)- K),0\}$ to make

$\max\{(X_2(\tau)- K),0\}+\max\{X_2(T)-X_2(\tau),0\}$ or give

$\max\{(X_2(\tau)- K),0\}$ + [multiple of {1-Rank of $X_2(\tau)$}]

Define a stopping time based on an occupation time under a prefixed level.

$$\tau = \inf\{t: \int_0^t \mathrm{I}\{\mathrm{X(u)}<\mathrm{M}_1\}\mathrm{du}>\mathrm{L}\}.$$

We use only X_2 here ($X \equiv X_2$).

Then, define a Payoff as in the above:

$\max\{(X_2(\tau)- K),0\}+\max\{X_2(T)-X_2(\tau),0\}$ or give

$\max\{(X_2(\tau)- K),0\}$ + [multiple of {1-Rank of $X_2(\tau)$}]

A recent research has proved that a probability density function of a stopping time is available for Ornstein-Uhlenbech process as well. So we can used Ornstein-Uhlenbech process in this framework, for defining a stopping time on an interest rate.

$$\tau = \inf\{t: \int_0^{t} \mathrm{I}\{\mathrm{X(u)<M_1}\}\mathrm{du>L}\}$$

$$P\{\int_0^{t} \mathrm{I}\{\mathrm{X(u)<M_1}\}\mathrm{du>L}\} = 1 - P\{\int_0^{t} \mathrm{I}\{\mathrm{X(u)<M_1}\}\mathrm{du<L}\}$$

$= 1 - P\{V(t) \le L\}$ where

$\mathrm{V(t)} = \int_0^{t} \mathrm{I}\{\mathrm{X(u)<M_1}\}\mathrm{du}$ is an occupation time process.

$V(t)$ is nondecreasing as a function of t.

Note that $\mathrm{P}\{\tau < \mathrm{t}\} = \mathrm{P}\{\mathrm{V(t)} \le \mathrm{L}\}$.

In general,

τ can be any as long as its probability distribution is known.

As long as the distribution of stopping time τ is known,

the probability distribution of

{Rank of $X_2(\tau)$ in time interval $[\tau, T]$})

can be given(calculated) and

it does not depend on the value of $X_2(\tau)$.

Comment 2: Applications .
Another situation for a second Example.
$X_1(t)$:economy of a country (say,US).
$X_2(t)$: currency exchange rate of Japanese Yen to US-dollar.
Time Interval [0,T]

Now set a boundary A that is higher than $X_1(0)$.

If X_1 hits the Level A (economy of the country US is recovered: Japanese Yen tends to become weaker against US Dollar;X2 higher), start, at the time τ of hitting, a stochastic corridor

(1-{Rank of $X_2(\tau)$ in time interval [τ, T]}) where $\tau=\inf\{t:X_1(t)>A\}$.

:$\{1\text{-}\mathrm{Rank}(X_2(\tau);[\tau,T], X_2)\}$.

Let the pay-off be **{a constant Yen amount)x**$\{1\text{-}\mathrm{Rank}(X_2(\tau);[\tau,T], X_2)\}$.

This pay-off will save an importing company in Japan by hedging against weak Japanese Yen.

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In order to design it for exporting Japanese company, we can replace (1-rank) with rank in the above setting.

Section 2. Value at Risk

: Statistical models.

: Shape of distribution.

Normality, Non-Normality and Nonparametric.

: iid (time-independent) and non iid(time-dependent).

Asia-Pacific Financial Markets 7: 305–319, 2000.

Statistical Methodologies for the Market Risk Measurement

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Abstract. This paper classifies statistical methodologies available for the market risk measurement. With the help of the weighted likelihood, a broad class of non-normal distributions, which are not generally considered so far, are applied to possibly hetero-scedastic financial variables. The approach is compared with popular procedures such as GARCH and J. P. Morgan's using daily data of 12 financial variables.

1. Introduction

In recent years quantitative risk management has become a central activity of financial institutions. Among various risk measures, value-at-risk (VaR) is a widely used measure of market risk in a portfolio of financial variables. Beyond the simple i.i.d. normal model that was first proposed by BIS, present-day risk managers are strongly interested in those methodologies that are able to capture the characteristics possibly overlooked by the simple model.

The VaR measurement is to estimate the lower $\alpha\%$ point of the probability distribution which portfolio value increments are assumed to follow. Let V_t be the value of the portfolio at time t. Given the information available at time t, such as the past values of financial variables, the probability of $V_{t+1} - V_t$ being above the estimated VaR needs to be (100-α)%. A typical value of α is 1, but theoretically can be any positive number less than 50. The essential part is to identify the statistical model that best approximates the mechanism generating the past data.

7. Summary

This paper considers a taxonomy of statistical methodologies available for the VaR measurement. It is summarized in the 3 by 2 matrix of {normal, non-normal, non-parametric} × {i.i.d., time-dependent}. Resorting to weighted likelihood approach, we show that there is no empty cell in the matrix. A broad class of non-normal distributions as well as the empirical distribution can be modified to fit into possibly hetero-scedastic cases.

Transformation models are introduced as flexible families of probability distributions whose parameters quantify deviations from the normality.

We also show that the logic of back-testing has some problems. Back-testing is a useful benchmark but should not be fussed over too severely.

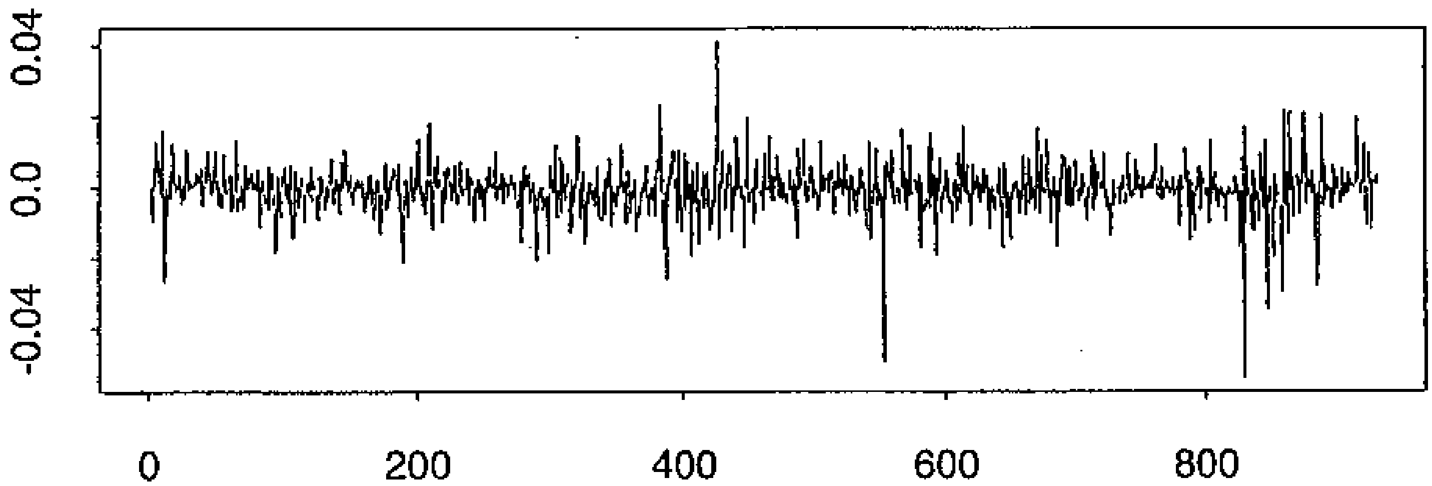


Figure 1. YEN/USD daily data: time series plot of log-ratios.

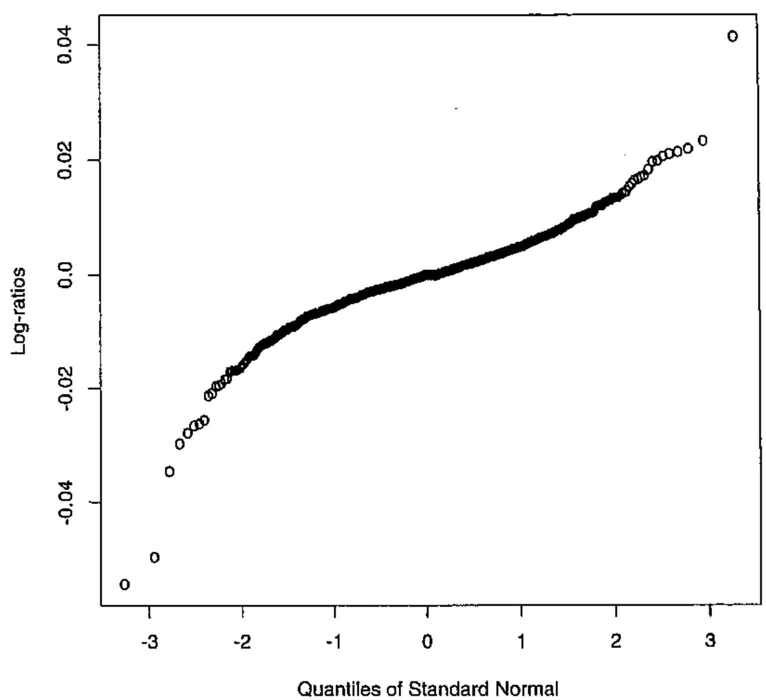


Figure 2. YEN/USD daily data: normal probability plot of log-ratios.

2. The I.I.D. Normal Model

Let $S_{i,t}; i = 1, \ldots, n$ be the values of n assets in portfolio at time t. For each factor asset, the rate of return $X_{i,t} = (S_{i,t+1} - S_{i,t})/S_{i,t}$ is the building block of the estimation as follows. The rate of return is approximately the log-ratio $Y_{i,t} = \log S_{i,t+1} - \log S_{i,t}$. Given the investment ratios $a_i; i = 1, \ldots, n$ with $\sum_{i=1}^{n} a_i = 1$, the rate of return of the portfolio is written as $X_t = \sum_{i=1}^{n} a_i X_{i,t}$ and

$$V_{t+1} - V_t = V_t X_t = V_t \sum_{i=1}^{n} a_i X_{i,t} .$$

The starting point is then making a graph of the past returns as in Figure 1 where the log-ratios of YEN/USD daily data are plotted against time. Once we assume that daily returns are independent and identically distributed (i.i.d.), the normal probability plot such as Figure 2 is useful to identify the shape of the probability distribution.

Under the i.i.d. normal assumption, $X_{i,t}, X_{i,t-1}, \ldots$ are assumed to be i.i.d. normal random variables for each i. The mean $\mu_i = E(X_{i,t})$ and the covariance $\sigma_{ij} = \mathrm{Cov}(X_{i,t}, X_{j,t})$ are therefore time-independent. Then,

$$V_{t+1} - V_t = V_t \sum_{i=1}^{n} a_i X_{i,t} \sim N(V_t \mu_p, (V_t \sigma_p)^2), \tag{1}$$

where $\mu_p = \sum_{i=1}^{n} a_i \mu_i$ and $\sigma_p^2 = \sum_i \sum_j a_i a_j \sigma_{ij}$. Given the value of V_t, the increment is also a normal random variable. The $\alpha\%$ point of the portfolio increment is easily computed as $V_t(\mu_p - z_\alpha \sigma_p)$ using the $\alpha\%$ quantile z_α of the standard normal distribution.

Given a sample $X_{i,t-1}, \ldots, X_{i,t-T}$ over T time periods, the usual estimates of the mean vector $\mu = (\mu_1, \ldots, \mu_n)^\mathsf{T}$ and the variance-covariance matrix $\Sigma = [\sigma_{ij}]$ are the sample mean $\hat{\mu}$ and the sample variance-covariance matrix $\hat{\Sigma}$; i.e.,

$$\hat{\mu}_i = \frac{1}{T} \sum_{s=1}^{T} X_{i,t-s}, \quad \hat{\sigma}_{ij} = \frac{1}{T} \sum_{s=1}^{T} (X_{i,t-s} - \hat{\mu}_i)(X_{j,t-s} - \hat{\mu}_j).$$

Once we have these estimates, the VaR of a portfolio with arbitrary value of investment ratios can be economically estimated from the Equation (1).

Normality provides "summability" of estimated parameters.
i.e. estimated parameter of portfolio can be obtained from the estimates of each assets included in the portfolio.

3. Non-Normal I.I.D. Models

On the frequency distributions of log-ratio of financial variables such as stock indices and currency exchange rates, the heaviness of tails and the strong kurtosis have been frequently reported. For example, Figure 2 suggests that the normal model does not fit well for very small and large returns of YEN/USD exchange rate. Such observations stimulate the trials of fitting heavier tailed distribution than normal law; e.g., the stable distributions, logistic distribution, and t-distribution. Some trials have expanded toward a construction of the pricing scheme for options with the underlying variables whose log-ratios obey heavier tailed distribution than normal.

For example, we assume that the cumulative distribution function (CDF) G of X_t is given by

$$G(x) = F\left(\frac{x - m}{s}\right),$$

where $F(x) = 1/(1 + e^{-x})$ is the CDF of logistic distribution, m is the location parameter, and s is the scale parameter. When the maximum likelihood estimates (MLE) $\hat{m}$ and $\hat{s}$ are used, the estimated $\alpha\%$ point is given by $\hat{m} - \hat{s} \cdot F^{-1}(0.01)$.

Figure 3 shows the VaR estimates for daily YEN/USD exchange rates by i.i.d. logistic model along with i.i.d. normal model. The VaR estimates by both models behave very similarly but ones by logistic model are slightly more flexible, that is, have more up and down movements.

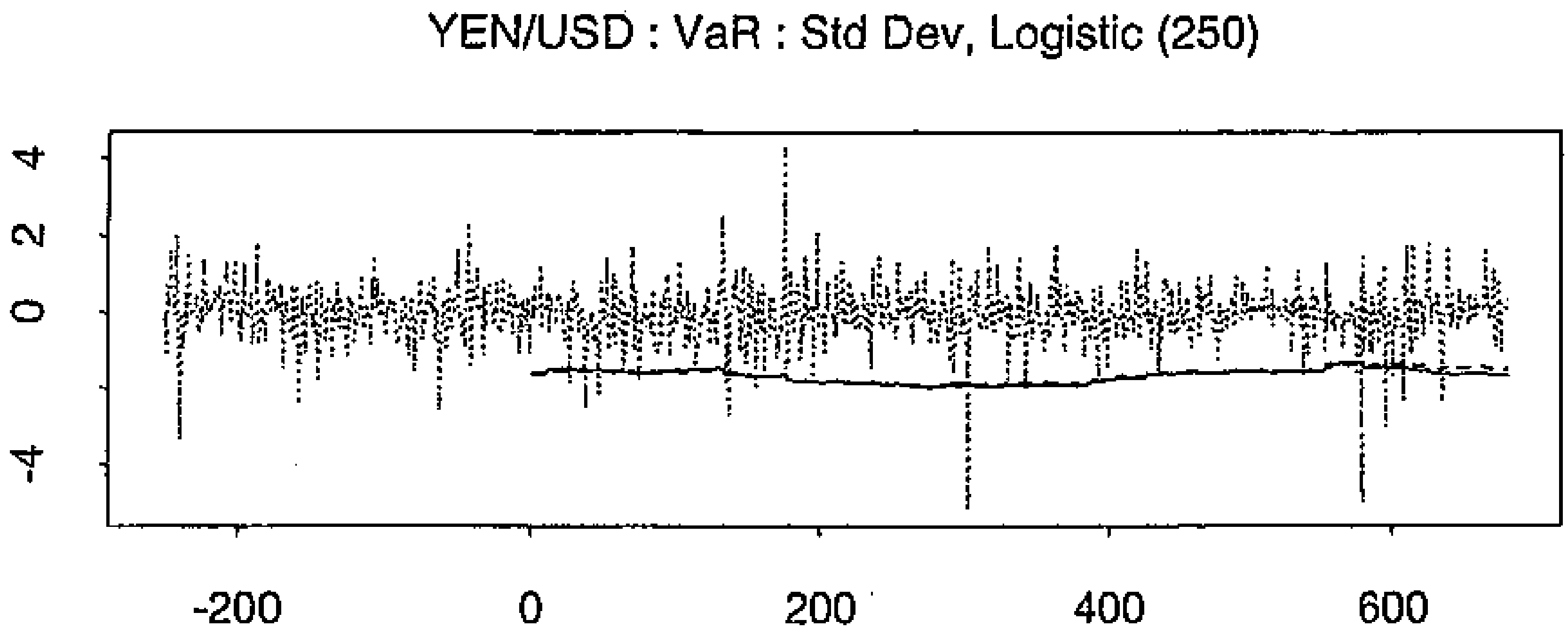


Figure 3. YEN/USD daily data: 1% VaR estimates by i.i.d. normal model (solid line) and i.i.d. logistic model (broken line) along with log-ratios.

3.1. TRANSFORMATION MODELS

A class of models called *transformation models* (Miura and Tsukahara, 1993) is very useful for dealing with deviations from the normality. The CDF G of a typical transformation model has the following form:

$$G(x) = h\left(F\left(\frac{x - m}{s}\right) : \theta\right),$$

where $h(\cdot : \theta)$ is a strictly increasing and continuous function on $[0, 1]$ with $h(0 : \theta) = 0$ and $h(1 : \theta) = 1$. Figures 5 and 7 are examples of h. The parameter θ

controls the deviation from F, the standard normal CDF for our case, and adjusts to peculiarities in each time period such as heavy-tails or skewness. Given the estimates $\hat{\theta}$, $\hat{m}$ and $\hat{s}$, the $\alpha\%$ point is estimated by $\hat{m} - \hat{s} \cdot F^{-1}(h^{-1}(0.01 : \hat{\theta}))$. We discuss more detail about transformation models in Section 4.

3.2. NONPARAMETRIC APPROACH

One can also use the empirical frequency distribution to estimate the VaR. This approach is often called *historical simulation*. Let $I\{X_s \leqslant x\}$ be the indicator function that takes one if $X_s \leqslant x$ or zero otherwise. Given $X_{t-1}, \ldots, X_{t-T}$, we define the empirical CDF $\hat{F}$ as

$$\hat{F}(x) = \sum_{s=1}^{T} \frac{1}{T} I\{X_{t-s} \leqslant x\}$$

which is an estimate of the theoretical CDF F of the rates of return X_s. Under the i.i.d. assumption, $I\{X_s \leqslant x\}$ are i.i.d. Bernoulli random variables with the probability of occurrence $F(x)$. Therefore, $n\hat{F}(x)$ is a Binomial random variable and the MLE of $F(x)$, which is also unbiased, is $n\hat{F}(x)$. The estimate of $\alpha\%$ point is given by $\hat{F}^{-1}(\alpha/100)$.

A percentile point of the empirical CDF is very sensitive to a few outliers. In Figure 4, we see a period of relatively large volatility affects succeeding VaR estimates.

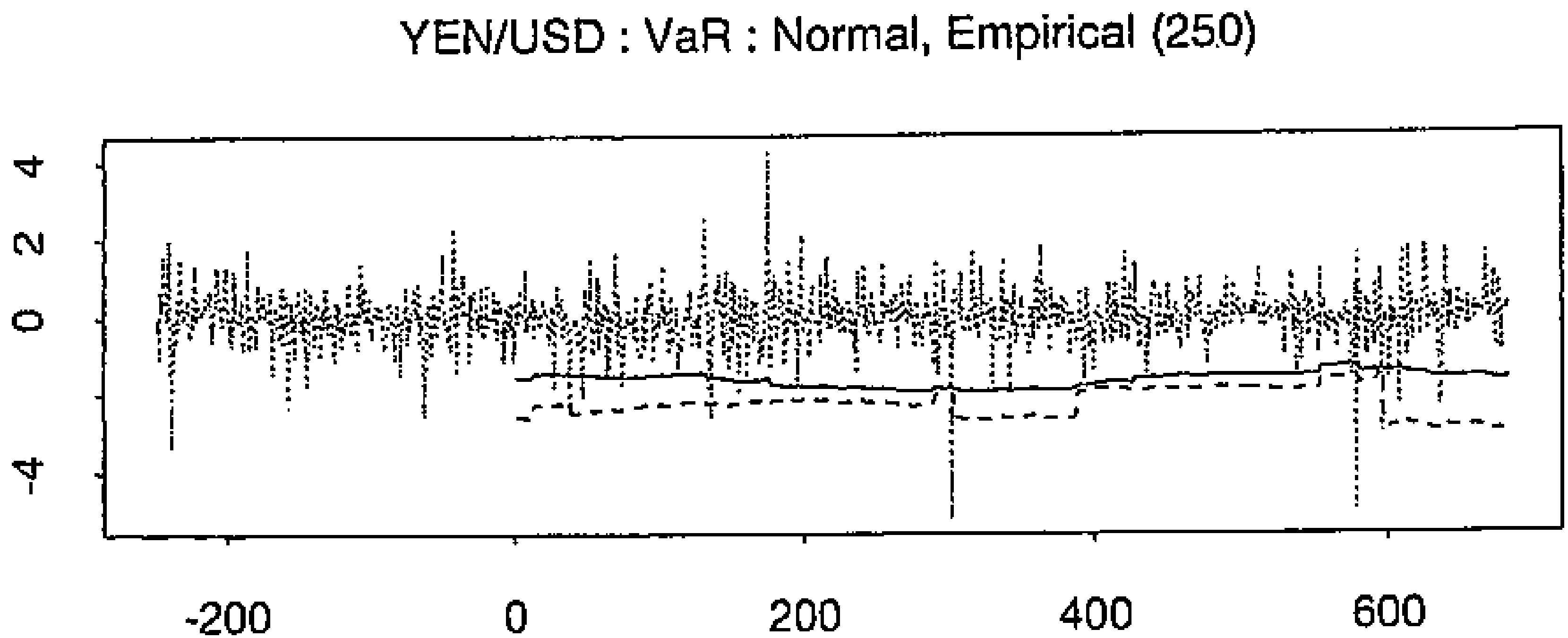


Figure 4. YEN/USD daily data: 1% VaR estimates by i.i.d. normal model (solid line) and empirical CDF model (broken line) along with log-ratios.

4. Examples of Transformation Models

Let us assume that an observation X_s follows CDF G where $G(\cdot)$ is of the form

$$G(\cdot) = h(F(\cdot) : \theta),$$

where $h(\cdot : \theta)$ is a strictly increasing and continuous function on [0, 1] for each θ with $h(0 : \theta) = 0$ and $h(1 : \theta) = 1$; i.e., $h(\cdot : \theta)$ is a distribution function on [0, 1] whose density is positive on [0, 1].

Let U be a uniform random variable on [0, 1]. X is regarded as a mapped value of U by G^{-1}, which in turn is a compounded mapping of h^{-1} and F^{-1}; i.e.,

$$X \equiv G^{-1}(U) = F^{-1}(h^{-1}(U : \theta)).$$

If F is the correct CDF, $h(\cdot)$ becomes the identical map and U is directly mapped by F^{-1}. The functional shape of h controls the deviation of G from F. When $F(\cdot)$ is normal, we aim to find a suitable family of functions $h(\cdot : \theta)$ so that $G(\cdot)$ is a non-normal, e.g., heavy-tailed or skewed CDF.

4.1. TRUNCATED POISSON COMPOUND

$$h(t:\theta) = \frac{\exp(\theta t) - 1}{\exp(\theta) - 1}, \quad \theta \in \mathbb{R}.$$

Note that $h(t:0) = t$ for $t \in [0, 1]$ by L'Hospital's rule. The transformation is expected to represent a skewness of distributions (Figures 5 and 6).

For $\theta > 0$, the function h is related to the maximum of the observations coming in when the number of observations N follows a truncated Poisson distribution with parameter θ; i.e., a Poisson distribution conditional on $\{N > 0\}$. If Y_i, $i = 1, \ldots, N$ are i.i.d. with distribution function F given N,

$$\begin{aligned} P\{\max Y_i < x\} &= \sum_{n=1}^{\infty} P\{\max\{Y_1, \ldots, Y_N\} < x | N = n\} P\{N = n\} \\ &= \sum_{n=1}^{\infty} F^n(x) \cdot \frac{\theta^n}{n!} \cdot \frac{e^{-\theta}}{1 - e^{-\theta}} \\ &= \left\{e^{\theta F(x)} - 1\right\} \cdot \frac{1}{e^{\theta} - 1}. \end{aligned}$$

The parameter θ indicates the *intensity* of underlying trading responses or flow of market information.

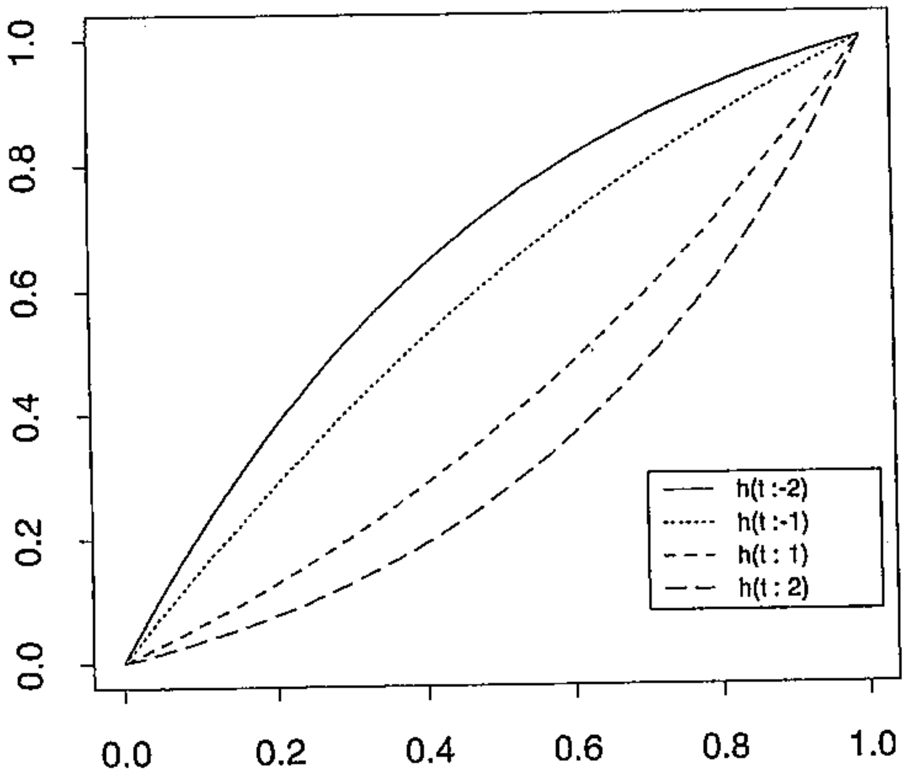
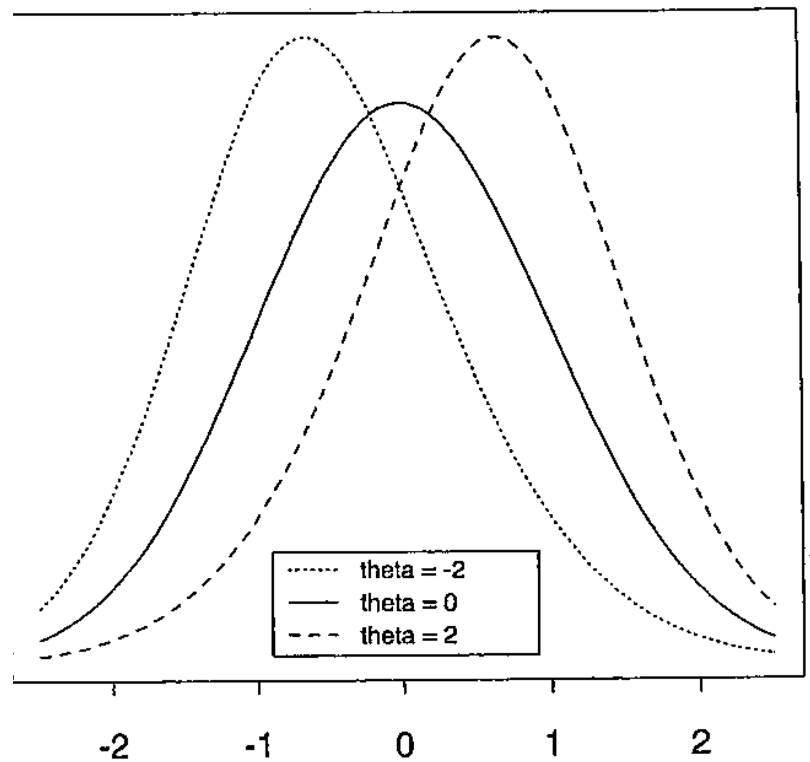


Figure 5. Truncated Poisson compound transformation.



6. Density functions of truncated Poisson compound models.

4.2. ANTI-SYMMETRIC BETA DISTRIBUTION FUNCTION

$$h(t:\theta) = \int_0^t \frac{1}{B(\theta,\theta)} s^{\theta-1}(1-s)^{\theta-1} ds, \quad \theta > 0,$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is a Beta function. The choice $\theta = \alpha = \beta$ provides a family of anti-symmetric transformations around $1/2$, i.e.,

$$h(t:\theta) = 1 - h(1-t:\theta), \quad t \in [0, 1/2].$$

Note that $h(t:1) = t$ for $t \in [0,1]$. This transformation is expected to control the heaviness of tails as shown in Figures 7 and 8.

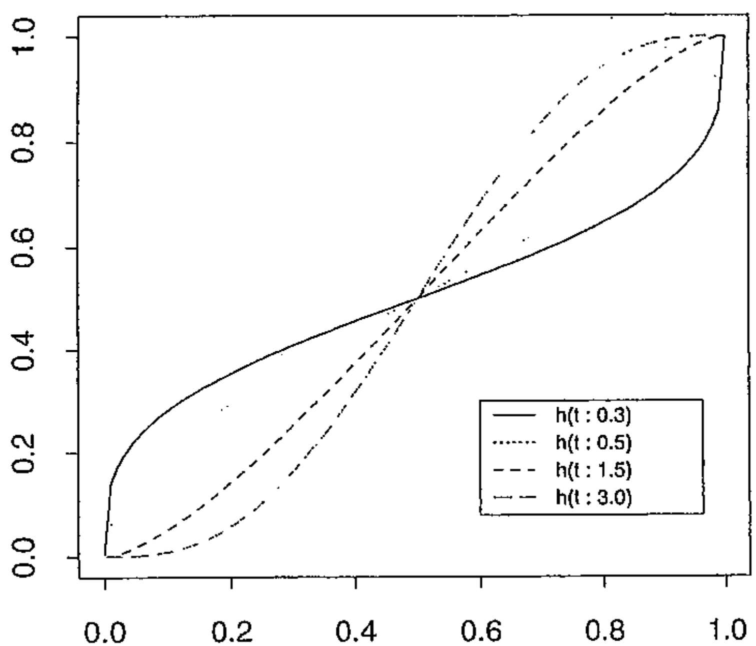


Figure 7. Anti-symmetric Beta transformation.

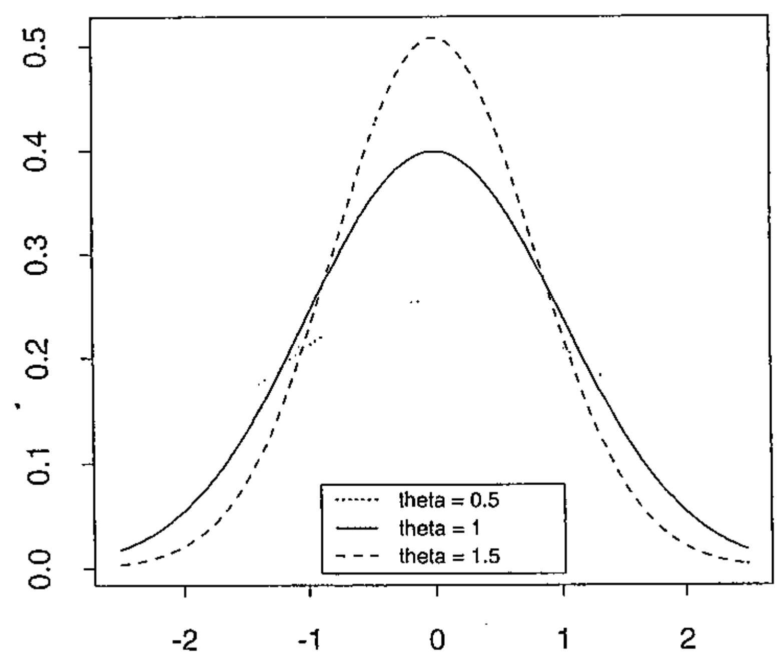


Figure 8. Density functions of anti-symmetric beta transformation.

Suppose that we have n i.i.d. observations $Y_1, \ldots, Y_n$ with common distribution function F. The transformation is related to the kth-smallest observation $Y_{(k)}$ as follows:

$$\begin{aligned} P\{Y_{(k)} < x\} &= P\{F(Y_{(k)}) < F(x)\} \\ &= P\{U_{(k)} < F(x)\} \\ &= \int_0^{F(x)} \frac{1}{B(k, n-k+1)} s^{k-1}(1-s)^{n-k} ds \,, \end{aligned} \tag{2}$$

where $U_{(k)}$ is the kth-*ordered statistics* of n i.i.d. Uniform random variables over [0, 1] and follows the Beta distribution with parameter $(k, n-k+1)$. The anti-symmetric Beta transformation $h(\cdot : \theta)$ corresponds to the circumstance that the *median* of observations is alway reported and thus $\theta \approx k \approx n-k+1$. The parameter θ controls the intensity of market information flow as before.

5. Time-Dependent Models

Another way to look at the deviations from normality is time-dependent volatility. For example, a typical empirical distribution of simulated Gaussian data whose variances change stochastically has heavier tails than those of normal distribution. This *stochastic volatility* has been recognized in the VaR measurement of financial portfolios and schemes such as GARCH and *exponentially weighted mean-variance*, both of which extend the simple i.i.d. normal model, are incorporated to take care of these non-normalities and hetero-scedasticity of distributions (J. P. Morgan, 1995).

It is often claimed that daily rates of return have time-varying or stochastic volatilities for large number of assets. For example, time series plots such as Figure 1 are considered to support the evidence that there are periods of relatively high or low variations. In this section, we discuss a statistical model and a hybrid of statistical models and data analytic approach (Tukey, 1977) for possibly time-varying volatilities.

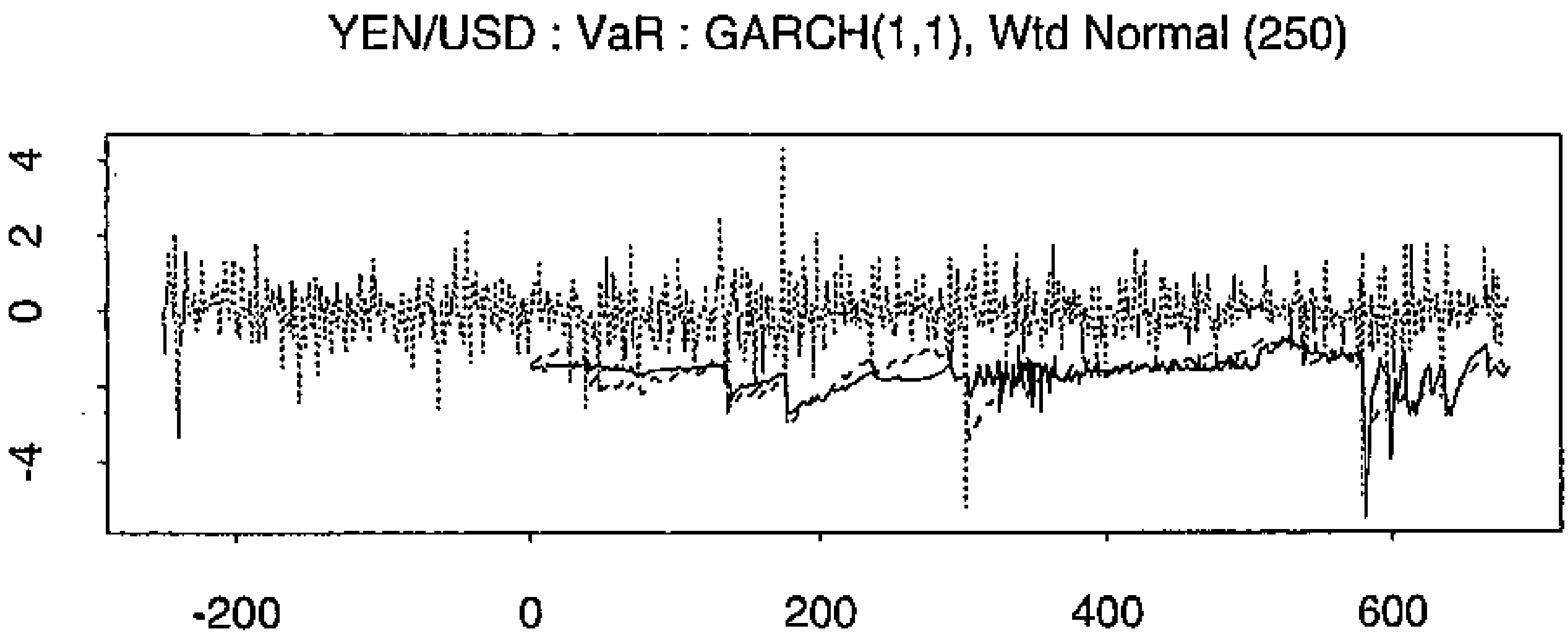


Figure 9. YEN/USD daily data: 1% VaR estimates by GARCH(1,1) model (solid line) and weighted normal model (broken line) along with log-ratios.

Figure 9 draws an example of the VaR estimates by univariate GARCH(1,1) model. The VaR estimates move rather radically as if they copied the rates of return themselves. We may ask whether the move is too sharp from a practitioner's point of view.

5.2. WEIGHTED LIKELIHOOD FUNCTIONS

The exponentially weighted mean and variance have been recognized as useful practical devices to catch up with time-varying volatilities in data (J. P. Morgan, 1995). For $j = 1, \ldots, T$, let

$$w_j = \frac{\lambda^j}{\sum_{j=1}^{T} \lambda^j}, \quad 0 < \lambda < 1. \tag{3}$$

Given $X_{t-1}, \ldots, X_{t-T}$, the exponentially weighted sample variance is computed as

$$\tilde{\sigma}^2 = \sum_{j=1}^{T} w_j (X_{t-j} - \tilde{\mu})^2,$$

where $\tilde{\mu} = \sum_{j=1}^{T} w_j X_{t-j}$ is the exponentially weighted sample mean. The J. P. Morgan's estimate of the $\alpha\%$ point is given by $\tilde{\mu} - z_\alpha \tilde{\sigma}$. As seen in Figure 9, the estimated VaR move more flexibly than the i.i.d. normal model but not so radical as GARCH(1,1). The parameter λ controls the roughness of the VaR estimates. In the above example, we set $\lambda = 0.94$. The VaR of a portfolio can be treated in a similar way as the i.i.d. normal model by replacing the sample mean and covariance with their weighted counterparts.

Here we show the J. P. Morgan's estimate is a special case of more general class of procedures. Let $f(\cdot|\theta)$, $\theta \in \Theta$ be a parametric family of density functions. We define the *weighted likelihood function* of X_{t-j}, $j = 1, \ldots, T$ by

$$L^e(\theta) = \Pi_{j=1}^T f(X_{t-j}|\theta)^{w_j} \tag{4}$$

and the *weighted log-likelihood function* by

$$l^e(\theta) = \sum_{j=1}^{T} w_j \log f(X_{t-j}|\theta)\,. \tag{5}$$

The latter is more convenient for numerical treatments.

When f is the normal family; that is,

$$f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,,$$

The estimates of μ and σ^2 which maximize (4) and (5) are, in fact, exponentially weighted mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$.

One can consider that the relative frequencies are now weighted by w_j's instead of 1 for the i.i.d. case so that the recent observations have more weight than the remote ones. This is to provide time-varying effect on the estimates by letting the observations to concentrate more on the recent values in order to take care of the recent changes in data. Here we do not construct a statistical model for time-dependent structures in data, but treat the dependence by weighting. In a sense, this is a hybrid of formal statistical models and data analytic approach.

The advantage of introducing weighted likelihood is that we no longer have to restrict ourselves to the normal law. Instead, we can let f be logistic or a transformation model. Given the parameter estimates based on (5), the $\alpha\%$ point of the distribution can be estimated as in Sections 3 and 4. In Figure 10, we compare the VaR estimates of weighted truncated Poisson compound model and weighted normal model. Though both estimates behave very similarly, a difference appears when the rate of return jumps upward. The Poisson model accepts these jumps as skewness of the distribution, whereas the normal model captures them as large volatility in both tails. As a result, the normal model may overestimate the VaR when symmetry in the frequency distribution is missing. Any symmetric model including GARCH shows the same reaction to large upward jumps.

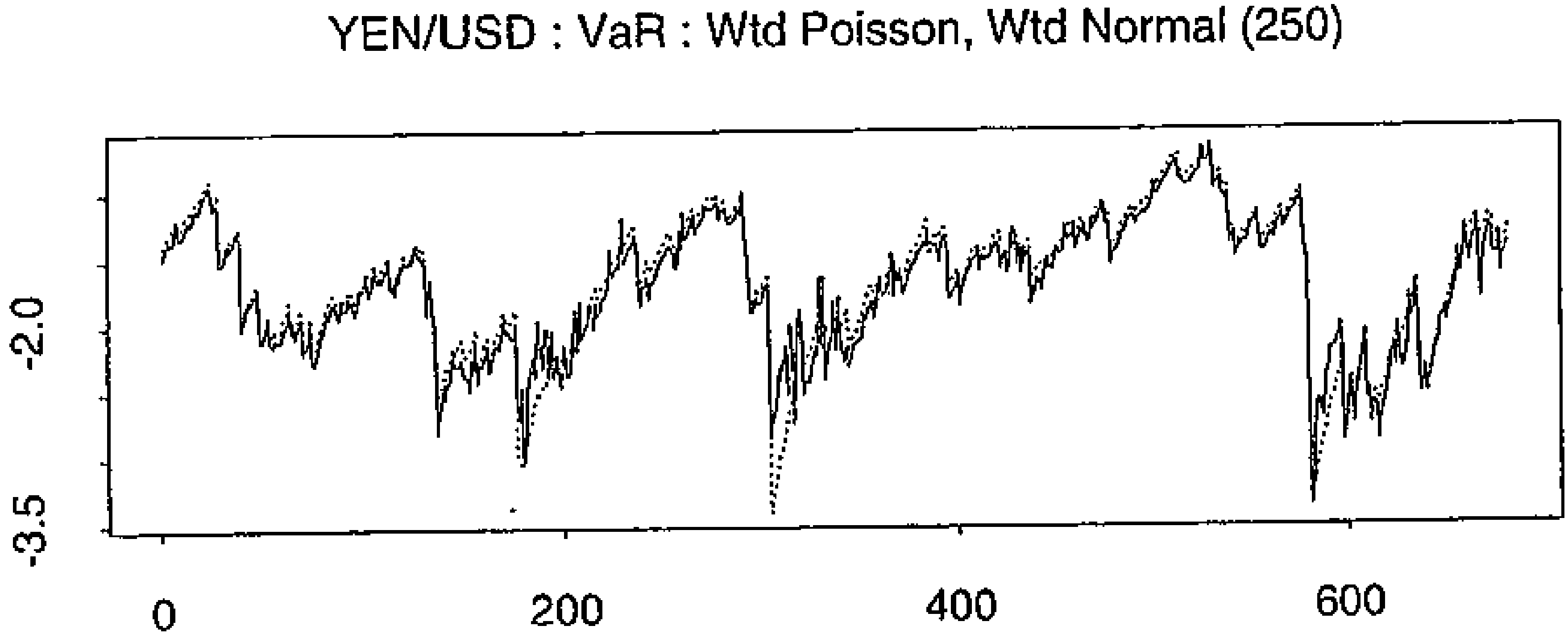


Figure 10. YEN/USD daily data: 1% VaR estimates by weighted Poisson compound model (solid line) and weighted normal model (broken line).

5.3. WEIGHTED EMPIRICAL CDF

The idea of weighted likelihood is also applied to the nonparametric approach which were based on empirical CDF. Here we introduce the weighted empirical CDF that assigns a weight to each observation depending on which time point the observation is taken. Given a set of weights such as (3), the weighted empirical CDF is defined by

$$\hat{F}_e(x) = \sum_{j=1}^{T} w_j I\{x : X_{t-j} \leq x\}.$$

As we have seen in Section 3.2, the usual empirical distribution functions are based on i.i.d. observations and put equal weight $w_j \equiv 1/T$ on each observation.

This construction will form a basis for the nonparametric approach to take care of time-dependent structure in data. As we see in Figure 11, the VaR estimates based on the weighted empirical CDF are rather conservative but more flexible to change than the i.i.d. empirical CDF model, since effects of relatively large observations die rapidly as time goes.

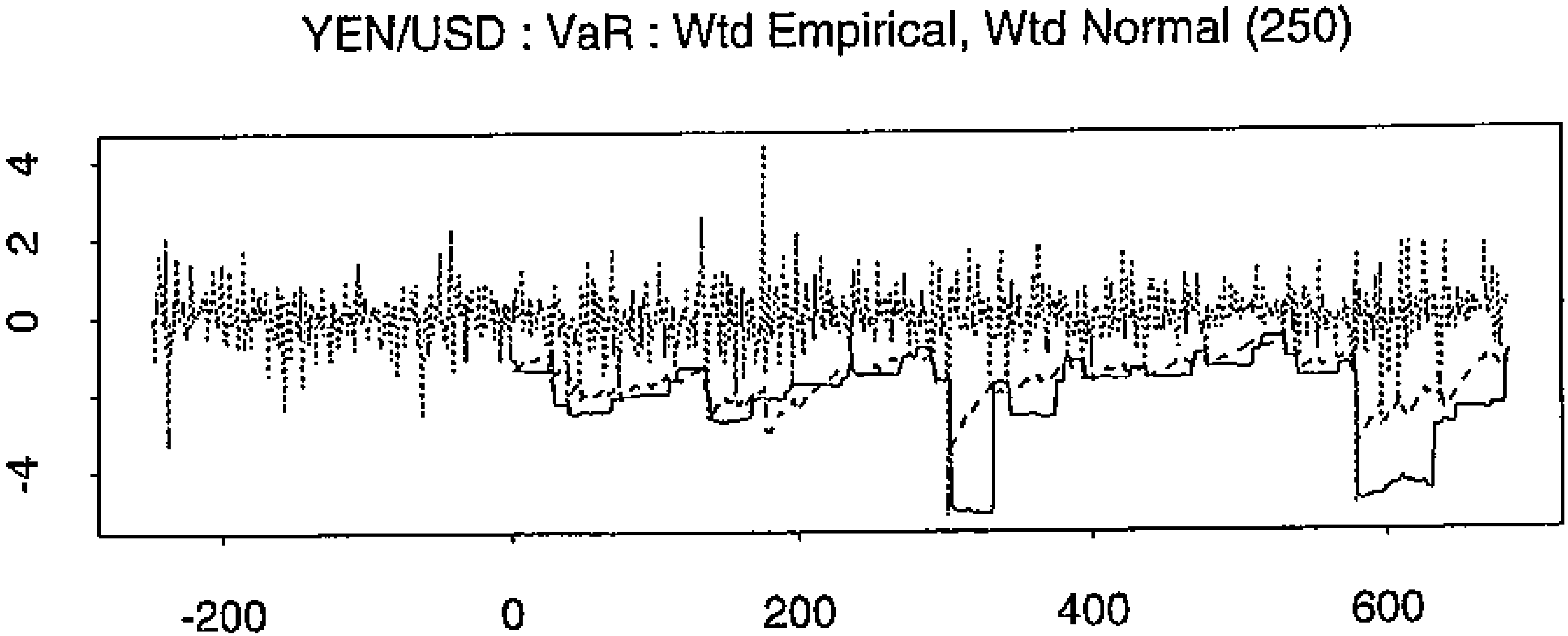


Figure 11. YEN/USD daily data: 1% VaR estimates by weighted empirical CDF model (solid line) and weighted normal model (broken line).

図 6-1

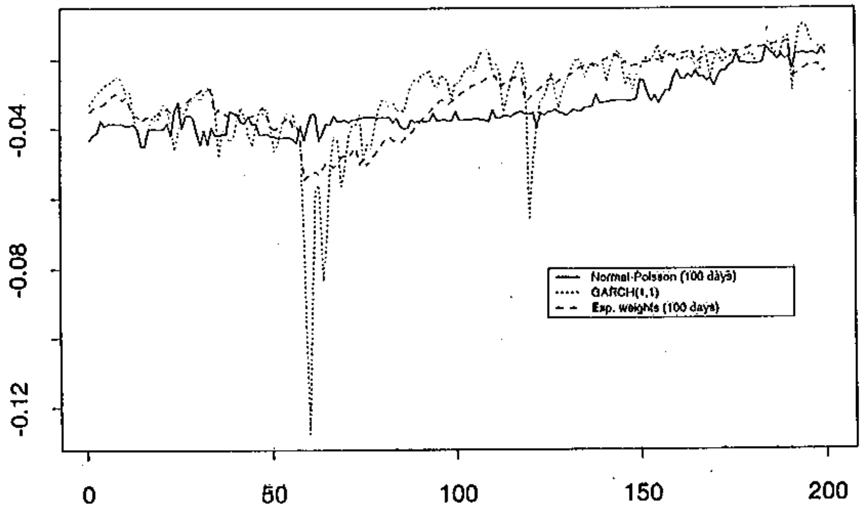


Figure 8: The 1st percentiles for TOPIX data

図 6-2

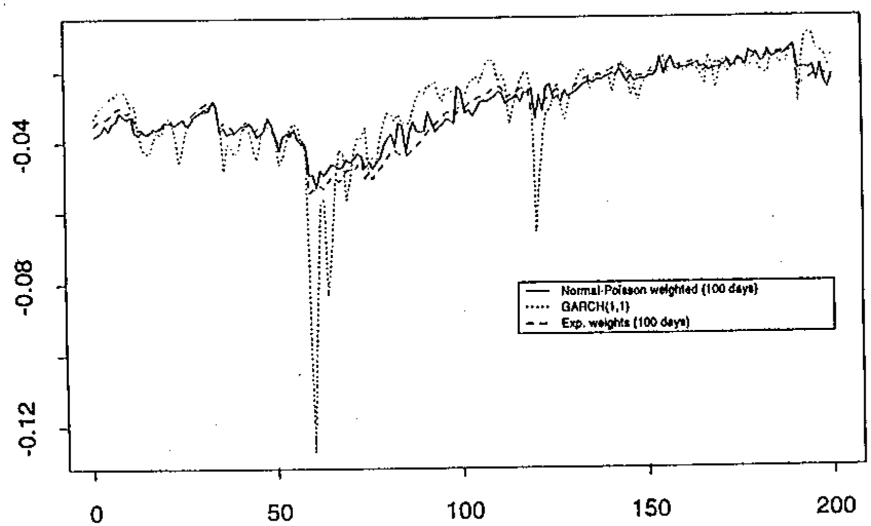


Figure 9: The 1st percentiles for TOPIX data

Equally weighted
compound Poisson

Look at
the solid Line

Exponentially weighted
Compound Poisson

6. Back-Testing

Let Q_t be the estimated 1% quantile of X_t at time t. Specifically or not, Q_t is a function of the past observations $X_{t-1}, \ldots, X_{t-T}$ and , $\ldots$, $Q_{t-1}, Q_t, Q_{t+1}, \cdots$ may well be dependent random variables. The idea of back-testing is to judge the goodness of a VaR estimation procedure by counting the frequency of the events $\{X_t < Q_t\}$. The frequency is expected not to be far off 1%.

We tested 12 financial variables that consist of 3 stock indices (DAX, S&P500, TOPIX), 2 exchange rates (GMM/USD, YEN/USD), 6 interest rates (LIBOR6m, USDSwap2yr, FIBOR6m, GMMSwap2yr, TIBOR6m, YENSwap2yr), and GOLD. For each of these variables, we applied the i.i.d. normal model(N), the i.i.d. logistic model(L), the i.i.d. empirical CDF model(E), the weighted normal model(WN), the weighted truncated Poisson compound model(WP), the weighted empirical CDF model(WE), and GARCH(1,1)(G). The period of observations is from 3 June 1992 to 31 July 1995. To estimate VaR on one day, 250 past log-ratios were used. The result is summarized in Table I.

Table I. Observed frequencies (%) of 1% tail events for 12 financial variables.

	N	L	E	WN	WP	WE	G
DAX	1.90	1.02	0.73	1.61	1.76	2.20	1.46
SP500	1.90	1.90	0.73	2.34	2.20	2.34	2.34
TOPIX	1.76	1.61	1.32	2.34	2.34	3.22	1.76
GMM/USD	1.61	1.32	0.88	2.34	2.49	2.05	2.20
YEN/USD	2.64	2.49	0.73	2.93	2.78	2.64	3.22
LIBOR6m	1.46	1.46	0.59	2.64	1.90	1.76	1.61
USDSwap2yr	2.20	1.46	1.02	2.20	2.49	2.34	2.05
FIBOR6m	2.34	3.95	0.59	3.22	2.64	2.34	3.07
GMMSwap2yr	1.76	1.61	0.59	1.17	1.32	2.15	1.17
TIBOR6m	5.27	6.59	1.76	3.95	4.10	3.07	4.98
YENSwap2yr	3.66	3.37	2.34	2.78	2.78	2.93	2.64
GOLD	1.61	1.76	0.59	2.05	2.20	2.05	2.34

The returns of short interest rates typically have rare but enormous outliers. For such variables, the i.i.d. models, except the empirical CDF, show some tendency of underestimating VaR. The i.i.d. empirical CDF model is consistently conservative.

Remark (worry) on Accuracy

Let us define $U_t = I\{X_t < Q_t\}$. The rationale behind the back-testing argument is that $\{U_t\}$ are expected to behave as independent Bernoulli trials with probability of occurrence equal to 0.01. However, it is not hard to imagine that the ideal situation does not hold in many cases. First, since successive Q_t are estimated on overlapping samples of returns, the independence of U_t is dubious. Second, even if Q_t is an unbiased estimate of the 1% point, the unbiasedness of U_t, i.e., $E\{U_t = 1\} = P\{X_t < Q_t\} = 0.01$, is not assured. Contrary to the conjecture in Hull and White (1998), U_t may be biased because of the estimation error in Q_t. We see this by the following simple example.

Suppose that $X_t \sim N(\mu, \sigma^2)$ and $Q_t \sim N(\eta, \gamma^2)$. We assume Q_t is an unbiased estimate of the 1% quantile of X_t so that $\eta = \mu - 2.33\sigma$. Then,

$$P\{X_t < Q_t\} = P\left\{\frac{X_t - \mu}{\sigma} < \frac{Q_t - \mu}{\sigma}\right\}$$
$$= P\left\{\frac{X_t - \mu}{\sigma} < \frac{Q_t - \eta}{\gamma}\frac{\gamma}{\sigma} - 2.33\right\}.$$

Since $(Q_t - \eta)/\gamma$ is a standard normal variate, one cannot always expect $P\{X_t < Q_t\} \approx 0.01$. The effect of the term is negligible only when the estimation error of Q_t is very small; that is, γ is far smaller than σ. The implication of this example is that a 'good' estimate of VaR with unbiasedness and small variance may produce a good performance of back-testing, but the other direction is not always guaranteed.

They can be safe if $\frac{Q - \eta}{\lambda}\frac{\lambda}{\sigma}$ is zero.

But, it is not so, I am afraid.

Testing if Log ratios are i.i.d.

BDS statistics

$\log(S_{t+1}/St)$, t=1,2,….

The following definition of BDS statistics is borrowed from this paper.

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Technical Note

AN INVESTIGATION OF CHAOS IN THE RL-DIODE CIRCUIT USING THE BDS TEST

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The time series to be analysed $(X_t : t = 1, 2, ..., T)$ is used to form the so-called N-histories

$$X_t^N = (X_t, X_{t+1}, ..., X_{t+N-1})$$

Each N-history can be considered to be point in an N-dimensional space, where N is called the embedding dimension. These N-histories can be used to define a correlation integral

$$C_N(e) = \frac{2}{T_N(T_N - 1)} \sum_{t<s} \sum I_e(X_t^N, X_s^N),$$

where $T_N = T - N + 1$, and I_e is the indicator function of the event

$$| X_{t+i} - X_{s+i} |< e, \; i = 0, 1, ..., N - 1.$$

i.e. $I_e(X_t^N, X_s^N)$ is unity if $| X_t^N - X_s^N |< e$ and zero otherwise. The correlation integral, $C_N(e)$, can be interpreted as an estimate of the probability that X_t^N and X_s^N are within a distance e. Given this interpretation, we can see that under the independence hypothesis

$$C_N(e) \longrightarrow C_1(e)^N, \text{ as } T \longrightarrow \infty$$

$$C_N(e) \longrightarrow C_1(e)^N, \text{ as } T \longrightarrow \infty$$

holds. That is, $P(| X_{t+i} - X_{s+i} |< e)$, $(i = 0, 1, ..., N-1)$ is, due to independence, equal to $\prod_{i=1}^{N-1} P(| X_{t+i} - X_{s+i} |< e)$, which is estimated by $C_1(e)^N$ as the variables are identically distributed (Brock,*et al.*, 1991 and Chappell, *et al.*, 1996). Thus, the BDS statistic reduces to

$$W_N(e) = [\sqrt{T}(C_N(e) - C_1(e)^N]/\hat{\sigma}_N(e),$$

where $\hat{\sigma}_N(e)$ is an estimate of the standard devision under the null hypothesis. The distribution of $W_N(e)$ converges to a standard normal with expectation zero and a variance unity, as T approaches infinity. Thus, one can now calculate the statistic that has a standard normal asymptotic distribution under the independence hypothesis. If the absolute values of the test statistic are large, the null hypothesis of IID (randomness) is to be rejected. The critical values reported by Brock, *et al.*(1991) for significance levels of 0.05 and 0.01 are 2.22 and 3.40 respectively.

図２－４

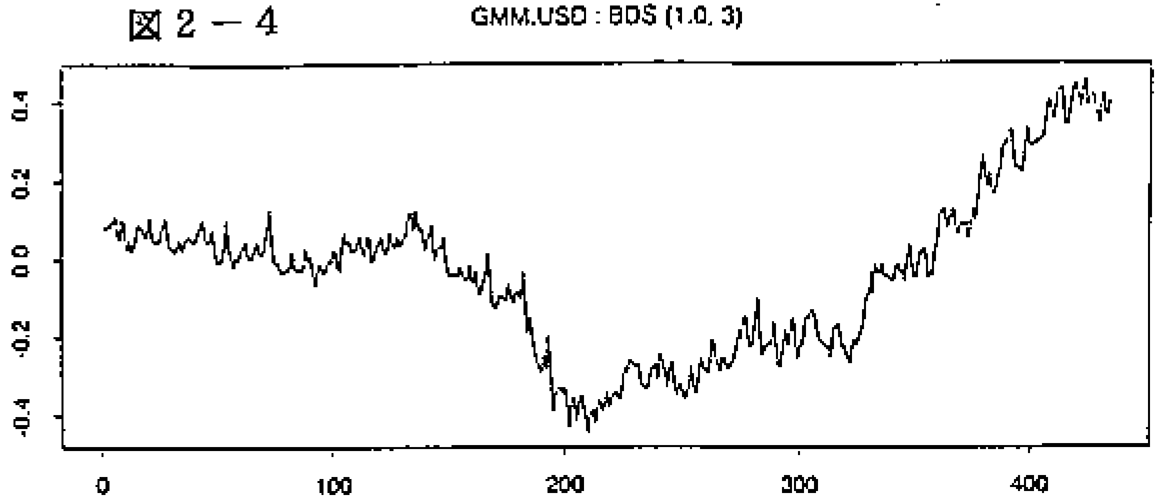

GMM.USD : BDS (1.0, 3)
0.4
0.2
0.0
-0.2
-0.4
0
100
200
300
400

図２－５

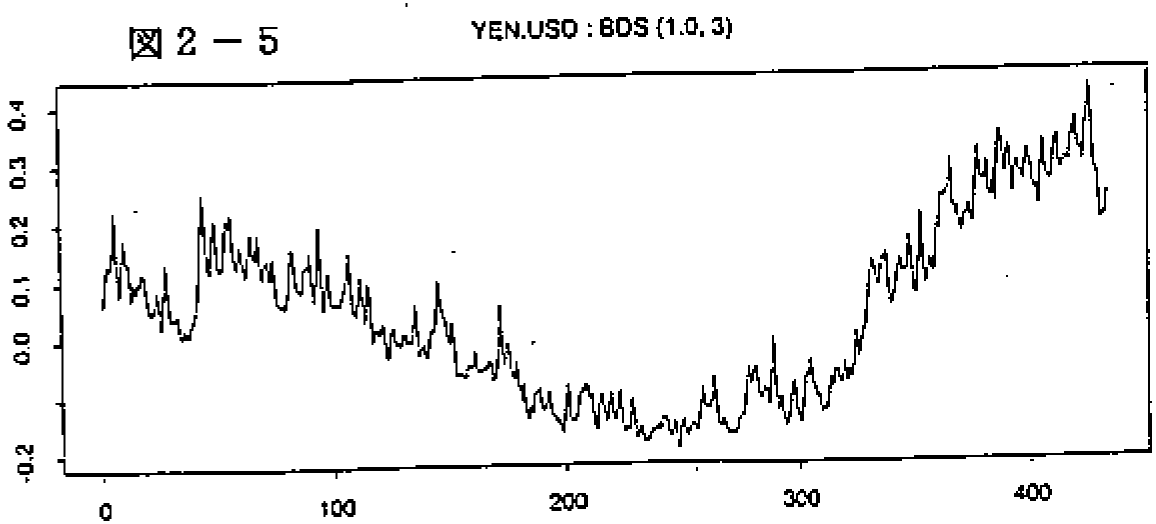

YEN.USD : BDS (1.0, 3)
0.4
0.3
0.2
0.1
0.0
-0.2
0
100
200
300
400

図2－1 TOPIX : BDS (1.0, 3)

図2－2 DAX : BDS (1.0, 3)

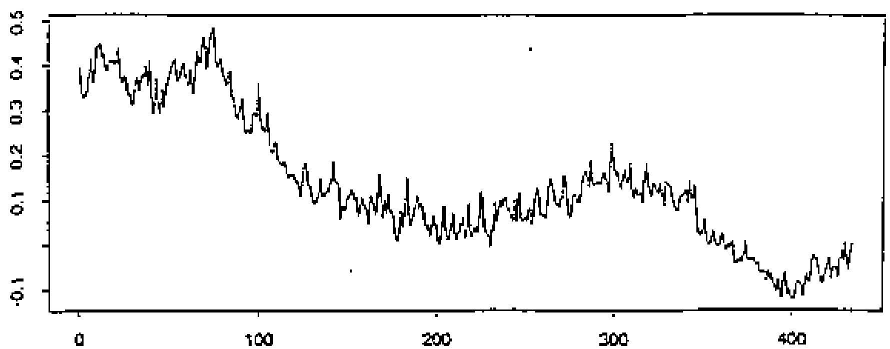
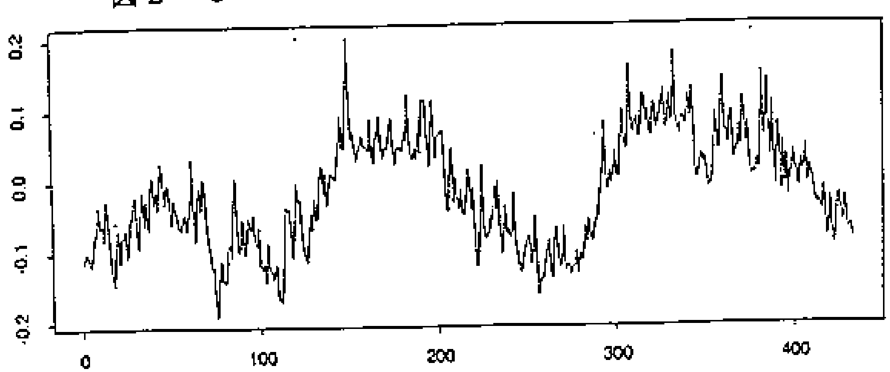


図2－3 SP500 : BDS (1.0, 3)



TOPIX(Japanese stock index) far above 0, but less than 1.65

SP500 arround 0

Section 3. Commodity linked Bond

Miura & Yamauchi (1998)

Probability of Default.

Linear SDE and its Solution.

Asia-Pacific Financial Markets **5:** 129–158, 1998.

The Pricing Formula for Commodity-Linked Bonds with Stochastic Convenience Yields and Default Risk

RYOZO MIURA[1] and HIROAKI YAMAUCHI[2]

Abstract. At the maturity, the owner of a commodity-linked bond has the right to receive the face value of the bond and the excess amount of spot market value of the reference commodity bundle over the prespecified exercise price. This payoff structure is an important characteristic of the commodity-linked bonds.

In this paper, we derive closed pricing formulae for the commodity-linked bonds. We assume that the reference commodity price and the value of the firm (bonds' issuer) follow geometric Brownian motions and that the net marginal convenience yield and interest rate follow Ornstein–Uhlenbech processes. In the appendix, we derive pricing formulae for bonds which are the same as the above commodity-linked bonds, except that the reference commodity price in the definition of the payoff at the maturity is replaced by the value of a special asset which depends on the convenience yield.

Key words: bond pricing, commodity-linked bond, convenience yield, default probability, PDE.

Pay-off of commodity bond.

Min{ V_T, F+max{$S_T - K, 0$}}

where

V_t , t $\in$[0, T], is the value of a firm which issues this bond.

F is a face value of bond

which the firm pays to the bond holder at time T.

S_t is the spot price of the underlying commodity.

K is a level such that a coupon amount max{$S_T - K, 0$}

will be added to F at time T.

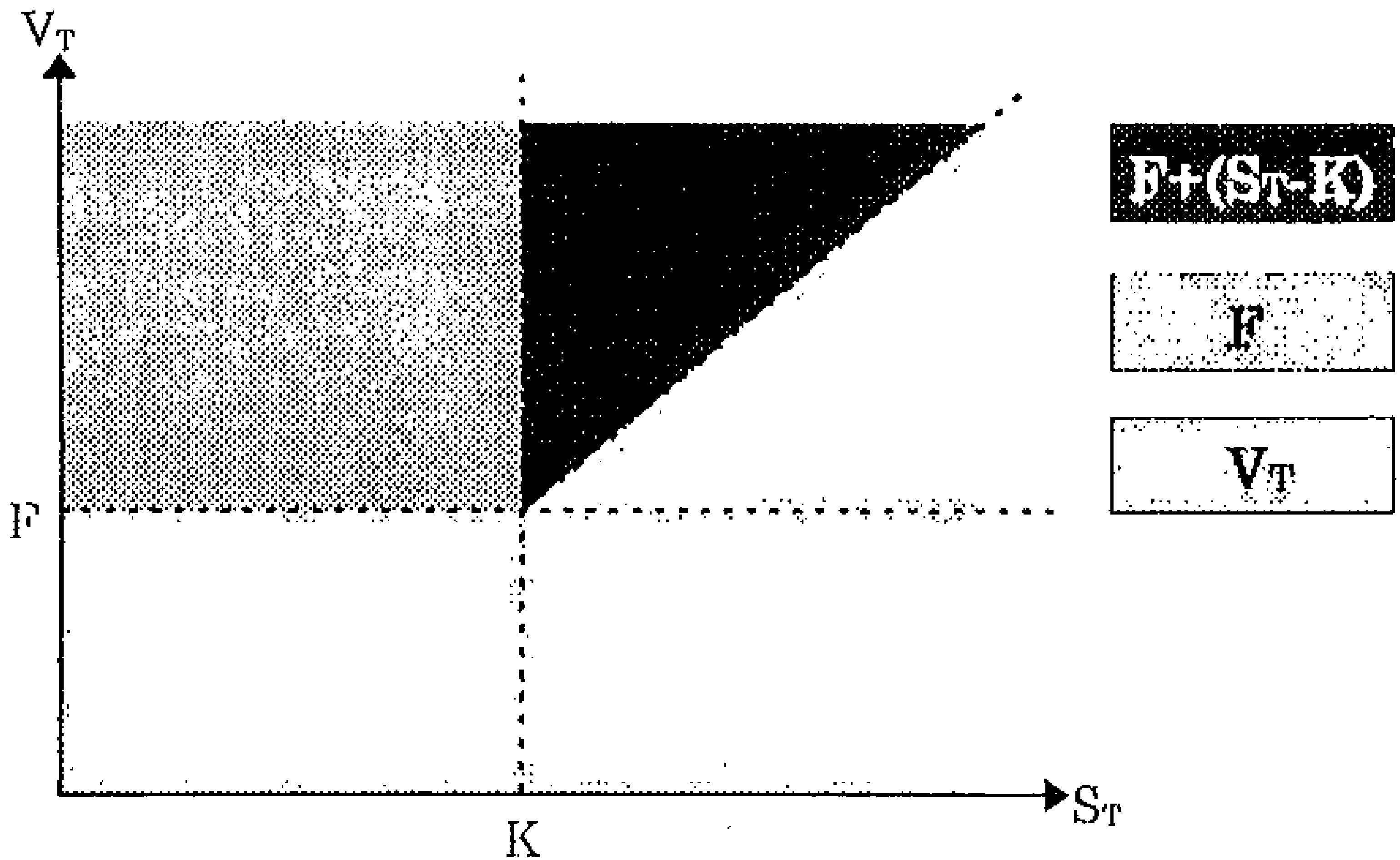


Figure 1. Payoff chart at the maturity.

condition.

Let S, V, and δ be stochastic processes. S_t is the spot price of the commodity, V_t denotes the value of the issuer (or the value of the firm), and δ_t represents the instantaneous net marginal convenience yield rate. We assume that S, V, and δ satisfy following stochastic differential equations (in short, SDE):

$$\frac{\mathrm{d}S}{S} = \alpha_S \cdot \mathrm{d}t + \sigma_S \cdot \mathrm{d}W_S \tag{1}$$

$$\frac{\mathrm{d}V}{V} = \alpha_V \cdot \mathrm{d}t + \sigma_V \cdot \mathrm{d}W_V \tag{2}$$

$$\mathrm{d}\delta = k(\mu_\delta - \delta)\mathrm{d}t + \sigma_\delta \cdot \mathrm{d}W_\delta \ , \tag{3}$$

where W_S, W_V, and W_δ are the standard Wiener processes and their correlation are such that $\mathrm{d}W_S \cdot \mathrm{d}W_V = \rho_{SV}\mathrm{d}t$, $\mathrm{d}W_S \cdot \mathrm{d}W_\delta = \rho_{S\delta}\mathrm{d}t$, and $\mathrm{d}W_V \cdot \mathrm{d}W_\delta = \rho_{V\delta}\mathrm{d}t$.

1.2. CONVENIENCE YIELD

The owner of the commodity has the rights (this is an option) to decide how he/she will treat the commodity; sell, lend, or store it, or even consume it. As for the consumption-use commodities such as crude oil or copper, the owner may consume it for his/her own manufacturing activities, or he/she may also store it for his/her future consumption or future sell-out. The owner of the futures contracts or the other contingent claims, however, does not have this rights because of lack of storage until the maturity.

Futures contracts

Spot price S_t and Futures price F_t.

In theories for no-arbitrage markets,

$E[S_T | S_t] = e^{r(T-t)} S_t = F_t$ if r is constant.

But if the commodity is ....(Oil in 1992, for example)

What Does *Backwardation* Mean?

A theory developed in respect to the price of a futures contract and the contract's time to expire. Backwardation says that as the contract approaches expiration, the futures contract will trade at a higher price compared to when the contract was further away from expiration. This is said to occur due to the convenience yield being higher than the prevailing risk free rate.

1.1. REVIEWS

Schwartz (1982) introduced a general framework for pricing commodity-linked bonds where (1) the reference commodity price follows a geometric Brownian motion and the interest rate is constant. He also covered in his framework the three other cases where (2) the commodity price and the bond price (i.e., the interest rate is stochastic) follow geometric Brownian motions, (3) the commodity price and the value of the firm (bond's issuer) follow geometric Brownian motions and the interest rate is constant, and (4) the interest rate behaves stochastically as an extension to the case. There he obtained the closed pricing formulae of commodity-linked bonds for the first three cases (1), (2) and (3). Defaults at the time of the maturity of the contingent claim (or, the bonds) of the issuing firms were considered in (3), where a pricing formula was derived. But he did not derive any closed pricing formula for the case (4). In his paper there was no discussion about the convenience yields.

Gibson and Schwartz (1990) is the first to consider the stochastic convenience yields for the bond pricing model. They derived the partial differential equation for the price functions of the assets defined as functions of spot commodity price and the net marginal convenience yield. They estimated parameters for the behavior of the net marginal convenience yield from market data, and calculated numerically the futures prices of the commodity.[1] Bjerksund (1991) derived a closed pricing formula for the commodity contingent claims where the commodity price follows a geometric Brownian motion, the net marginal convenience yield follows an Ornstein–Uhlenbech process, and the interest rate is constant. He did not consider the default of the issuing firms at the maturity of the commodity contingent claims. Gibson and Schwartz (1993) utilized Bejerksund's (also two other parties') pricing formula and Black's (1976) formula to fit the market prices of the crude oil futures options. Since our concerns in the present paper are the mathematical pricing formulae for the commodity contingent claims, we do not further refer their fitting results. They were able to calculate numerically the present prices of the commodity-linked bonds, but did not derive a closed analytical pricing formula.

1.3. OUR RESULTS

In this paper, we take the approach of Gibson and Schwartz (1990, 1993), Bjerksund (1991) to express the price change of the reference commodity in relation to its convenience yield. In Appendix C, we drive a pricing formula for a special derivative. The underlying asset of this derivative itself is a derivative security, which is seen in Bjerksund (1991), that consists of a commodity and the continuously reinvested net marginal convenience yield. These two appraoches reflect two ways of treatments, that we could take, for the pricing of the commodity contingent claims. The latter one uses the value of the ownership of the commodity as its underlying variables which receive the total expected return derived from its price changes and net marginal convenience yield. Then, we see that the resulting pricing formula does not explicitly depend on the parameters related to the movements of the convenience yield. On the other hand, the former one uses the market price of the commodity where the owner of the commodity contingent claim cannot receive the convenience yield deriving from the ownership of the commodity, but receives the total expected return from its price changes.

The pricing formula in the former case includes the parameters related to the convenience yield. By using this formula, we draw several graphs of the bond prices and the default probabilities. The default occurs when the total payoff to the bond holder exceeds the value of the issuer at the maturity. The figures for the default probabilities provide us useful information to the bond issuer/holder in regard to the risk management.

Note in Proof

After the final revision of this paper toward this publication, we came to know the work of K.R. Miltersen and E.S. Schwartz (1997) 'Pricing of Options on Commodity Futures with Stochastic Term Structure of Convenience Yields and Interest Rates'. Publications from Department of Management, School of Business and Economics, Odense University. Their paper develops a model for pricing options on commodity futures in the presence of stochastic rates as well as stochastic convenience yields.

Derivation of Closed from pricing formula

of Commodity linked Bonds.

Using the Standard PDE approach explained last week.

Case : assuming Interest rate r is constant.

The case of stochastic interest rate r is shown in section 4 of the paper.

2. Closed Pricing Formula for the Commodity-Linked Bonds $B(S_t, V_t, \delta_t, \tau)$

In this section, we derive the pricing functions of the commodity-linked bonds, $B(S_t, V_t, \delta_t, \tau)$. To start with, we define our stochastic variables and derive the PDE. Then we obtain the closed pricing formula for the commodity-linked bonds that satisfies the derived PDE with its payoff at the maturity as the boundary condition.

condition.

Let S, V, and δ be stochastic processes. S_t is the spot price of the commodity, V_t denotes the value of the issuer (or the value of the firm), and δ_t represents the instantaneous net marginal convenience yield rate. We assume that S, V, and δ satisfy following stochastic differential equations (in short, SDE):

$$\frac{\mathrm{d}S}{S} = \alpha_S \cdot \mathrm{d}t + \sigma_S \cdot \mathrm{d}W_S \tag{1}$$

$$\frac{\mathrm{d}V}{V} = \alpha_V \cdot \mathrm{d}t + \sigma_V \cdot \mathrm{d}W_V \tag{2}$$

$$\mathrm{d}\delta = k(\mu_\delta - \delta)\mathrm{d}t + \sigma_\delta \cdot \mathrm{d}W_\delta \ , \tag{3}$$

where W_S, W_V, and W_δ are the standard Wiener processes and their correlation are such that $\mathrm{d}W_S \cdot \mathrm{d}W_V = \rho_{SV}\mathrm{d}t$, $\mathrm{d}W_S \cdot \mathrm{d}W_\delta = \rho_{S\delta}\mathrm{d}t$, and $\mathrm{d}W_V \cdot \mathrm{d}W_\delta = \rho_{V\delta}\mathrm{d}t$.

Let the parameters be constant.

We postulate that the parameters α_S, α_V, κ, μ_δ, σ_S, σ_V, σ_δ, ρ_{SV}, $\rho_{S\delta}$, and $\rho_{V\delta}$ are constants. We assume in the above that $d\delta$ follows Ornstein–Uhlenbeck process that can take negative values. This is not a problem for the convenience yield, because our definition (3) is for the net marginal convenience yield. The net marginal convenience yield is defined by the differences that the gross convenience yield subtracted by the cost of carry, thus it sometimes takes negative values.

Let the interest rate r be constant.

The case of stochastic interest rate can be worked out.

It is done in the section 4 of this paper.

We also postulate that there is a risk free interest rate r and that this is a constant during the time interval from t to T. The length of this time interval is denoted by τ. In this paper, we assume that assets are infinitely divisible and that a short position is allowed. We also assume that there is no-arbitrage opportunity in the market.

Commodity-linked bonds have the payoff at the majority such that the owner of the bonds has right to receive, in the case of no default, in addition to the face value, the excess amount of the spot market price of the reference commodity over the prespecified exercise price. In the case where the default is considered, the payoff at the maturity is the minimum of either the payoff in the case of no default or the value of the issuer at the maturity. The total amount of the payment to the bond owner at the maturity is

$$\min[V_T, F + \max\{S_T - K, 0\}] . \tag{4}$$

F and K are constants, F is the face value of the bond and K is the prespecified exercise price of the reference commodity.

Assumptions on the parameters.

so that the assets in this market have distinguishable stochastics.

The following π's are defined in the next slide.

Also we assume that there are $N(N \geq 3)$ different assets in the market with price functions $B_i(S_t, V_t, \delta_t, \tau)$ for i-th asset, where $i = 1, 2, \cdots, N$, that have the same reference commodity. This is not an unrealistic assumption. Moreover, we postulate that for any choice of the three assets, the three vectors each of which consists of $\pi_S^i, \pi_V^i, \pi_\delta^i$ in the following equation (6) for $B_i(S_t, V_t, \delta_t, \tau)$ are linearly independent to each other: that is, the following matrices are non-singular for any choice of the three derivative assets.

$$\begin{bmatrix} \pi_S^i & \pi_V^i & \pi_\delta^i \\ \pi_S^j & \pi_V^j & \pi_\delta^j \\ \pi_S^k & \pi_V^k & \pi_\delta^k \end{bmatrix}, \tag{5}$$

where $i, j, k = 1, \cdots, N$, and $i \neq j$, $i \neq k$, and $j \neq k$.

Standard PDE approach (recall from the last week's lecture)

We used Ito stochastic defferentials.

Next, we derive the PDE for the pricing function of the commodity-linked bond, $B(S_t, V_t, \delta_t, \tau)$. By using Ito's lemma, we obtain the following equation.

$$\frac{\mathrm{d}B_i}{B_i} = \varphi_{B,i} \cdot \mathrm{d}t + \pi_S^i \cdot \mathrm{d}W_S + \pi_V^i \cdot \mathrm{d}W_V + \pi_\delta^i \cdot \mathrm{d}W_\delta \ , \tag{6}$$

where

$$\varphi_{B,i} = \frac{\left\{ \begin{array}{l} \frac{\partial B_i}{\partial S} S\alpha_S + \frac{\partial B_i}{\partial V} V\alpha_V + \frac{\partial B_i}{\partial \delta}\kappa(\mu_\delta - \delta) - \frac{\partial B_i}{\partial \tau} \\ +\frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial S^2} S^2\sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial V^2} V^2\sigma_V^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial \delta^2}\sigma_\delta^2 \\ +\frac{\partial^2 B_i}{\partial S \partial V} SV\sigma_S\sigma_V\rho_{SV} + \frac{\partial^2 B_i}{\partial S \partial \delta} S\sigma_S\sigma_\delta\rho_{S\delta} + \frac{\partial^2 B_i}{\partial V \partial \delta} V\sigma_V\sigma_\delta\rho_{V\delta} \end{array} \right\}}{B_i} \ ,$$

$$\pi_S^i = \frac{\partial B_i}{\partial S} \cdot \frac{S\sigma_S}{B_i} \ , \pi_V^i = \frac{\partial B_i}{\partial V} \cdot \frac{V\sigma_V}{B_i} \ , \pi_\delta^i = \frac{\partial B_i}{\partial \delta} \cdot \frac{\sigma_\delta}{B_i} \ .$$

We construct a portfolio W such that the portfolio consists of three different derivative assets and the commodity. We denote the weights of each assets in this portfolio as x_i $(i = 1, \cdots, 4)$ and the sum of these portfolio weights is equal to 1, i.e.,

$$\sum_{i=1}^{4} x_i = 1 \, .$$

Then the rate of return of the portfolio W is given by

$$\frac{\mathrm{d}W}{W} = x_1 \cdot \frac{\mathrm{d}B_1}{B_1} + x_2 \cdot \frac{\mathrm{d}B_2}{B_2} + x_3 \cdot \frac{\mathrm{d}B_3}{B_3} + x_4 \cdot \left(\frac{\mathrm{d}S}{S} + \delta_t \cdot \mathrm{d}t \right) \, . \tag{7}$$

This equation utilizes the property that the total rate of return of the owner of the reference commodity is the sum of the price changes of the commodity and its convenience yield.

Recall the arguments, from last week,

for "Riskless" portfolio and no-arbitrage.

We use a linear equation to derive "riskless" portfolio weights ;

variance(stand. dev.) terms be zero and portfolio weights sums to 1.

Also another linear equation to set "Drift" terms equal to interest rate r

By using the standard no-arbitrage argument, we obtain the following equations:

$$\begin{bmatrix} \varphi_{B,1} - r \\ \varphi_{B,2} - r \\ \varphi_{B,3} - r \\ \alpha_S + \delta_t - r \end{bmatrix} = \lambda_1 \cdot \begin{bmatrix} \pi_S^1 \\ \pi_S^2 \\ \pi_S^3 \\ \sigma_S \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} \pi_V^1 \\ \pi_V^2 \\ \pi_V^3 \\ 0 \end{bmatrix} + \lambda_3 \cdot \begin{bmatrix} \pi_\delta^1 \\ \pi_\delta^2 \\ \pi_\delta^3 \\ 0 \end{bmatrix}. \tag{8}$$

Note that λ_1, λ_2, and λ_3 are the market prices of risk for the commodity price, the value of the issuer, and the net marginal convenience yield, respectively. They are, logically at now, time dependent and vary with regard to the choice of the assets in the portfolio W. We show in the Appendix A that these do not depend on the choice of the assets. We assume that these λ's are constants in solving the following PDE (9). Note also that first, second, and third row of (8) are the equations for any derivative assets and the fourth row of (8) is the equation for the commodity itself.

From (8) we have the following PDE.

$$\frac{\partial B}{\partial S} S(r-\delta) + \frac{\partial B}{\partial V} V(\alpha_V - \lambda_2 \sigma_V) + \frac{\partial B}{\partial \delta} \{\kappa(\mu_\delta - \delta) - \lambda_3 \sigma_\delta\} -$$

$$- \frac{\partial B}{\partial \tau} - r \cdot B + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial S^2} S^2 \sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial V^2} V^2 \sigma_V^2 + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial \delta^2} \sigma_\delta^2 + \qquad (9)$$

$$+ \frac{\partial^2 B}{\partial S \partial V} S V \sigma_S \sigma_V \rho_{SV} + \frac{\partial^2 B}{\partial S \partial \delta} S \sigma_S \sigma_\delta \rho_{S\delta} + \frac{\partial^2 B}{\partial V \partial \delta} V \sigma_V \sigma_\delta \rho_{V\delta} = 0 .$$

Equation (9) is the PDE which every $B(S_t, V_t, \delta_t, \tau)$ must satisfy.

Our commodity linked bond has Pay-off .

Min{ V_T, F+max{$S_T - K, 0$}}.

Boundary condition $B(V_T, S_T, 0, 0) =$ Min{ V_T, F+max{$S_T - K, 0$}}

Next, we derive the closed form of the pricing function for the commodity-linked bonds $B(S_t, V_t, \delta_t, \tau)$ under the payoff function (4) at the bond maturity. This is done by applying Feynman–Kac Theorem (see Friedman, 1975, Chapter 6, Theorem 5.3). To calculate the expected value of the payoff function, where we write $[\tilde{S}_t, \tilde{V}_t, \tilde{\delta}_t]$ for the corresponding stochastic processes, we need to obtain the joint distribution function of $[\tilde{S}_t, \tilde{V}_t, \tilde{\delta}_t]$ based on PDE (9). The detailed derivation is shown in subsection 4.3.

From (9) we have the following SDE.

$$d\begin{bmatrix}\tilde{S}_t\\ \tilde{V}_t\\ \tilde{\delta}_t\end{bmatrix} = \begin{bmatrix}\tilde{S}_t(r-\tilde{\delta}_t)\\ \tilde{V}_t(\alpha_V-\lambda_2\sigma_V)\\ \kappa(\mu_\delta-\tilde{\delta}_t)-\lambda_3\sigma_\delta\end{bmatrix}\mathrm{d}t+\mathbf{G}\cdot d\begin{bmatrix}\tilde{Z}_{1,t}\\ \tilde{Z}_{2,t}\\ \tilde{Z}_{3,t}\end{bmatrix}, \tag{10}$$

where the $\tilde{Z}_{1,t}$, $\tilde{Z}_{2,t}$, $\tilde{Z}_{3,t}$ are another set of independent standard Wiener processes.

Note that the coefficients in PDE correspond to drift and variance in the corresponding SDE.

A choice of **G** is given by

$$\mathbf{G} = \begin{bmatrix} \tilde{S}_t \sigma_S & 0 & 0 \\ \tilde{V}_t \sigma_V \rho_{SV} & \tilde{V} \sigma_V \cdot c & 0 \\ \sigma_\delta \rho_{S\delta} & \sigma_\delta \cdot \bar{e} & \sigma_\delta \cdot f \end{bmatrix},$$

where $c = \sqrt{1-\rho_{SV}^2}$, $\bar{e} = \dfrac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\sqrt{1-\rho_{SV}^2}}$, $f = \sqrt{1 - \dfrac{(\rho_{V\delta} - \rho_{SV}\rho_{S\delta})^2}{1-\rho_{SV}^2} - \rho_{S\delta}^2}$.

The derivation of **G** is shown in Appendix B.

Because one of the drift term of SDE (10) is the product of $\tilde{S}_t$ and $\tilde{\delta}_t$, SDE (10) is *non-linear*. However, we make the following change of variables in order to transform this *non-linear* SDE to a *linear* SDE. Then let $\tilde{P}_t = \log \tilde{S}_t$ and $\tilde{J}_t = \log \tilde{V}_t$. By applying Ito's lemma, we have

$$d\begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \end{bmatrix} = \begin{bmatrix} -\tilde{\delta}_t + (r - \frac{1}{2}\sigma_s^2) \\ \alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2 \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \sigma_S & 0 & 0 \\ \sigma_V\rho_{SV} & \sigma_V \cdot c & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot \bar{e} & \sigma_\delta \cdot f \end{bmatrix} \cdot d\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \end{bmatrix}. \tag{11}$$

Substitute the following A,a, and G into the solution in the next slide.

$$A=\begin{pmatrix} 0,0,-1 \\ 0,0,0 \\ 0,0,-\kappa \end{pmatrix}, \quad a=\begin{pmatrix} r-\frac{1}{2}\sigma_S{}^2 \\ \alpha_V-\lambda_2\sigma_V-\frac{1}{2}\sigma_V{}^2 \\ \kappa\mu_\delta-\lambda_3\sigma_\delta \\ \end{pmatrix}$$

B is G

$$\Phi_t = e^{A(t-t_0)} = \begin{pmatrix} 1,0,\frac{1}{\kappa}(e^{-\kappa(t-t_0)}-1) \\ 0,1,0 \\ 0,0,e^{-\kappa(t-t_0)} \\ \end{pmatrix}, \Phi_t{}^{-1} = \begin{pmatrix} 1,0,\frac{1}{\kappa}(e^{\kappa(t-t_0)}-1) \\ 0,1,0 \\ 0,0,e^{\kappa(t-t_0)} \\ \end{pmatrix}$$

Note: Here, we need to solve the linear SDE.

Solution form is given in a text book(Arnold's,for example).

Or, you can just differentiate (stochastically) the solution,

then you will see that it satisfies the SDE.

By solving the stochastic differential Equations (19), we derive the joint probability density function of $[\tilde{P}_t, \bar{J}_t, \tilde{\delta}_t, \tilde{r}_t]$ for the time interval $[t_0, T]$ using theorem 8.2.2 in Arnold (1973, p. 129). By theorem 8.2.2, the SDE

$$d\tilde{\mathbf{Q}}_t = (\mathbf{A}(t) \cdot \tilde{\mathbf{Q}}_t + \mathbf{a}(t))dt + \mathbf{B}(t) \cdot d\tilde{\mathbf{Z}}_t$$

has the solution

$$\tilde{\mathbf{Q}}_t = \Phi_t(\mathbf{Q}_{t_0} + \int_{t_0}^{t} \Phi_s^{-1} \cdot \mathbf{a}(s) \cdot ds + \int_{t_0}^{t} \Phi_s^{-1} \cdot \mathbf{B}(s) \cdot d\tilde{\mathbf{Z}}_s) \tag{20}$$

with the initial value $\mathbf{Q}_{t_0}$, where

Note that the third term in a form of stochastic integral is a random variable with a zero-mean normal distribution.

Now by using Theorem 8.2.2 in Arnold (1973, p. 129), we can solve the SDE (11) and derive the joint distribution of $[\tilde{P}_t,\ \tilde{J}_t,\ \tilde{\delta}_t]$ for the time interval $[t_0, T]$. The solution of (11) is given by

$$\begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \end{bmatrix} = \begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \end{bmatrix} + \begin{bmatrix} \tilde{X}_t \\ \tilde{Y}_t \\ \tilde{Z}_t \end{bmatrix},$$

where $\pi = t - t_0$, $[\alpha(\pi), \beta(\pi), \gamma(\pi)]$ are deterministic functions such that

$$\begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \end{bmatrix} = \begin{bmatrix} P_{t_0} + \delta_{t_0}\frac{e^{-\kappa\pi}-1}{\kappa} + \left(r - \frac{\sigma_s^2}{2}\right)\pi + (\kappa\mu_\delta - \lambda_3\sigma_\delta)\frac{1-e^{-\kappa\pi}-\kappa\pi}{\kappa^2} \\ J_{t_0} + (\alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2)\pi \\ \delta_{t_0}e^{-\kappa\pi} + (\kappa\mu_\delta - \lambda_3\sigma_\delta)\frac{1-e^{-\kappa\pi}}{\kappa} \end{bmatrix},$$

and $[\tilde{X}_t,\ \tilde{Y}_t,\ \tilde{Z}_t]$ are jointly normally distributed. Their means are zero and the and $[\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t]$ are jointly normally distributed. Their means are zero and the variance-covariance matrix $\sum$ is given by

$$\sum = \begin{bmatrix} Var(\tilde{X}_t) & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{XY} & Var(\tilde{Y}_t) & \sigma_{YZ} \\ \sigma_{XZ} & \sigma_{YZ} & Var(\tilde{Z}_t) \end{bmatrix},$$

where

$$\begin{cases} Var(\tilde{X}_t) = \pi\left(\sigma_s^2 - 2\frac{\sigma_S\sigma_\delta\rho_{S\delta}}{\kappa} + \frac{\sigma_\delta^2}{\kappa^2}\right) + 2(1-e^{-\kappa\pi})\left(\frac{\sigma_S\sigma_\delta\rho_{S\delta}}{\kappa^2} - \frac{\sigma_\delta^2}{\kappa^3}\right) + \\ \qquad + (1-e^{-2\kappa\pi})\frac{\sigma_\delta^2}{2\kappa^3} \\ Var(\tilde{Y}_t) = \sigma_V^2\pi \\ Var(\tilde{Z}_t) = \frac{\sigma_\delta^2(1-e^{-2\kappa\pi})}{2\kappa} \\ \sigma_{XY} = \sigma_S\sigma_V\rho_{SV}\pi - \frac{\sigma_V\sigma_\delta\rho_{V\delta}\pi}{\kappa} + \frac{\sigma_V\sigma_\delta\rho_{V\delta}(1-e^{-\kappa\pi})}{\kappa^2} \\ \sigma_{XZ} = \frac{\sigma_S\sigma_\delta\rho_{S\delta}(1-e^{-\kappa\pi})}{\kappa} - \frac{\sigma_\delta^2(1-e^{-\kappa\pi})}{\kappa^2} + \frac{\sigma_\delta^2(1-e^{-2\kappa\pi})}{2\kappa^2} \\ \sigma_{YZ} = \frac{\sigma_V\sigma_\delta\rho_{V\delta}(1-e^{-\kappa\pi})}{\kappa} \end{cases}$$

The joint density function of $[\tilde{X}_t,\ \tilde{Y}_t,\ \tilde{Z}_t]$ is given by

$$f(\tilde{x}_t,\ \tilde{y}_t,\ \tilde{z}_t) = \frac{1}{(2\pi)^{3/2} \cdot \sqrt{\det \sum}} \cdot \exp\{-\tfrac{1}{2}\tilde{\mathbf{v}}^{\mathbf{T}} \textstyle\sum^{-1} \tilde{\mathbf{v}}\},$$

where

$$\tilde{\mathbf{v}} = \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \\ \tilde{z}_t \end{bmatrix}.$$

Then we calculate the present value of the expected value of the payoff function at the maturity (see Figure 1 for the payoff chart at the maturity). The solution of the PDE (9) with the boundary condition (i.e., payoff) (4) is given by

$$B(S_t,\ V_t,\ \delta_t\ ,t) = E\left[\min\{\tilde{V}_T,\ F + \max(\tilde{S}_T - K, 0)\} \cdot e^{-r\tau} \,\middle|\, \begin{array}{ccc} \tilde{S}_t & = & S_t \\ \tilde{V}_t & = & V_t \\ \tilde{\delta}_t & = & \delta_t \end{array} \right]$$

$$
\begin{aligned}
B(S_t,\ V_t,\ \delta_t\ ,t) &= E\left[\min\{\tilde{V}_T,\ F+\max(\tilde{S}_T-K,0)\}\cdot e^{-r\tau}\ \middle|\ \begin{array}{rcl}\tilde{S}_t &=& S_t\\ \tilde{V}_t &=& V_t\\ \tilde{\delta}_t &=& \delta_t\end{array}\right]\\
&= E\left[\min\{e^{\beta(\tau)+y_T},\ F+\max(e^{\alpha(\tau)+x_T}-K,0)\}\cdot e^{-r\tau}\ \middle|\ \begin{array}{rcl}\tilde{S}_t &=& S_t\\ \tilde{V}_t &=& V_t\\ \tilde{\delta}_t &=& \delta_t\end{array}\right]\\
&= F\cdot\exp\{-r\tau\}\cdot\int_H^\infty\int_{-\infty}^R f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t+\\
&+\exp\{\beta(\tau)-r\tau\}\cdot\int_{-\infty}^H\int_{-\infty}^R \exp\{y_t\}f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t+\\
&+\exp\{\beta(\tau)-r\tau\}\cdot\int_{-\infty}^Q\int_R^\infty \exp\{y_t\}f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t+\\
&+(F-K)\cdot\exp\{-r\tau\}\cdot\int_Q^\infty\int_R^\infty f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t+\\
&+\exp\{\alpha(\tau)-r\tau\}\cdot\int_Q^\infty\int_R^\infty \exp\{x_t\}f_{XY}(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t
\end{aligned}
\tag{12}
$$

where $f_{XY}(x_t,\ y_t)$ is the marginal density function of x_t and y_t and $\{H|y_T = \log\ F-\beta(\tau)\}$, $\{R|x_T = \log\ K-\alpha(\tau)\}$, and $\{Q|y_T = \log\{F+\exp\{\alpha(\tau)+x_T\}-K\}-\beta(\tau)\}$.

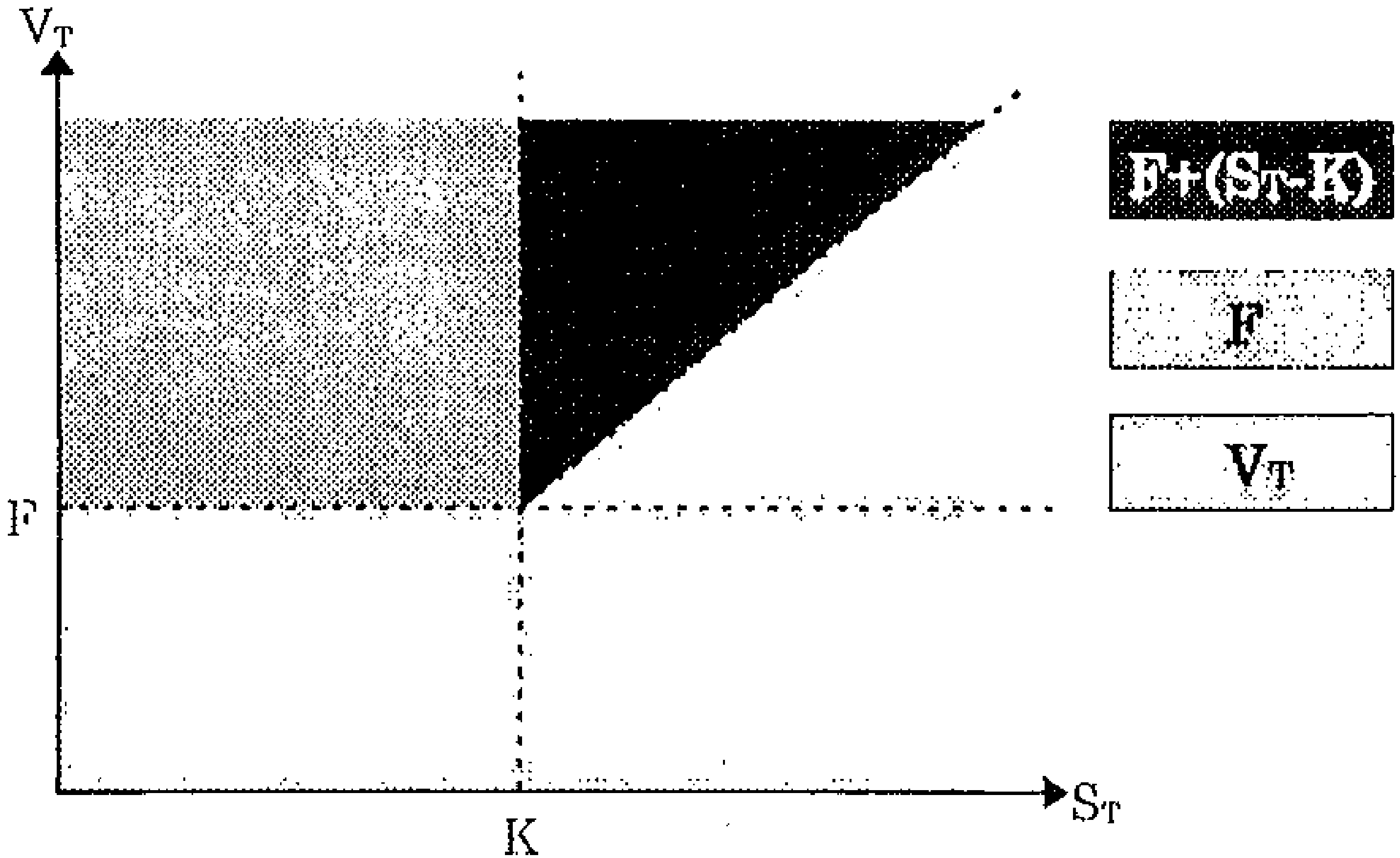


Figure 1. Payoff chart at the maturity.

Area :

$V_T > F + \max\{S_T - K, 0\}$ or $V_T > F + \max\{S_T - K, 0\}$,

with $S_T > K$ or $S_T < K$.

Numerical aspects of

Bond prices and

Probability of default

3. Bond Prices and Default Probabilities

In this section we show several figures for the bond prices and the default probabilities as functions of parameters. We list a Mathematica's program for computing the above pricing function (12) in Appendix D. The default probabilities are given by the following formula, that is, we calculated the probability that the total payoff to the bond holder is equal to V_T, i.e., the light gray area in the $(S_T,\ V_T)$ domain of Figure 1.

$$\text{Default probability} = \int_{-\infty}^{H}\int_{-\infty}^{R} f(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t + \\ + \int_{-\infty}^{Q}\int_{R}^{\infty} f(x_t,\ y_t)\mathrm{d}x_t\,\mathrm{d}y_t\,. \tag{13}$$

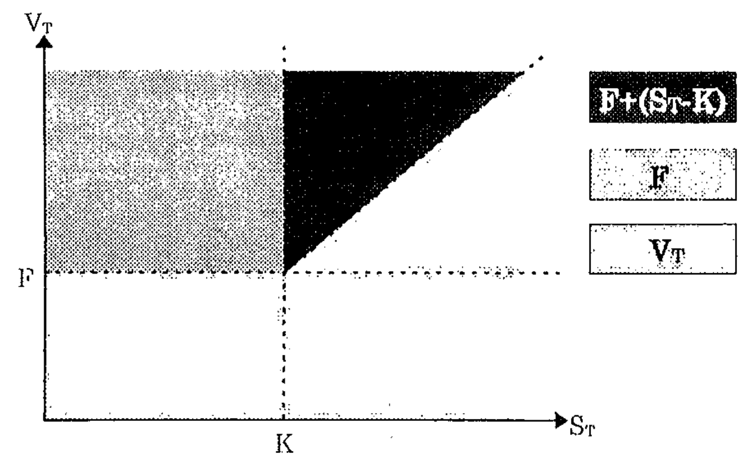


Figure 1. Payoff chart at the maturity.

When we use these formulae, we have to estimate each of paramters of the processes for S_t, V_t, δ_t and a constant r. For the commodity price S_t, and the

interest rate r, they are quoted in the market, thus we can observe directly each of them and estimate the parameters σ_S and the constant r. But it is difficult to observe the value of the issuer V_t, and the net marginal convenience yield δ_t, because they are not quoted or reported in the market. Therefore we have to estimate each of V_t and δ_t or to use some proxies instead. For the value of the issuer, one idea to estimate each of parameters α_V and σ_V of V_t is that: we can treat stock price of this issuer firm as a call option on the value of the issuer, then we squeeze out the value of the issuer firm from its stock price. Unfortunately to accomplish this idea is not an easy task because it is difficult to know the whole cash flow of the issuer.

In this section, the initial value of V_t and the parameters for the behavior of V_t are set rather subjectively. For the net marginal convenience yield, we recommend a simplified estimation procedure by Gibson and Schwartz (1990) or its revised scheme by Yamauchi (1998). Their methods use futures prices of the commodity with various delivery months. By isolating the differences between futures prices with neighboring delivery months and also by excluding interest rates effects for the futures prices, they approximately estimate one month or two months net marginal FORWARD convenience yield rate and might regard the AGGREGATED net marginal forward convenience yield rate as the net marginal covenience yield rate. Then they can be used to estimate each of parameters of δ_t .

The initial values required for the calculations of the price functions and default probabilities are set in the following way. The issuer of the commodity-linked bond has the business of producing and selling the commodity which is the underlying asset of that commodity-linked bond. The issuer wants to issue this bond with face value $F = 100$ and the maturity of 5 years. The strike price is set equal to the price of commodity at the time of issuance. The current interest rate is r

is 4% per year. The initial value of this issuer V_t is 200 which consists of this commodity-linked bond and the equity. Its expected growth ratio α_V is 2% per year and its volatility σ_V is 30% annually. These parameters of the value of the issuing firm are set so that the probability of $V_T < F$ is approximately 16% at the maturity. The current prices of commodity S_t is 20 and the price volatility σ_s is 39.2% per year. This volatility parameter is estimated from WTI crude oil prices data in NYMEX for the period from September 4, 1990 to June 20, 1994. To set the initial values of the parameters of the process of the net marginal convenience yields, we refer the detail to Yamauchi (1998). In his paper, he first calculated, from daily futures prices of different maturity, the daily values of 3 months net marginal convenience yield rates in a similar way to the one by Gibson and Schwartz (1990).

Then, based on these daily values, he estimated the parameters of the process δ. The whole estimation period is from September 4, 1990 to June 22, 1993. He divided this estimation period into two periods; from September 4, 1990 to June 22, 1991 and from June 23, 1991 to June 22, 1993. From the former period, the estimated parameters were such that: $\kappa = 19.122$, $\mu_\delta = 0.324$, and $\sigma_\delta = 1.3050$. From the latter period, $\kappa = 4.547$, $\mu_\delta = 0.021$, and $\sigma_\delta = 0.2673$. The estimated parameters suggest that the convenience yields of crude oil at the former period showed a very wild movements. During the latter period, the convenience yields seem to be relatively stable. In this paper, we set two situations, namely situation A and B. We use the estimated parameters from the former period for situation A and the latter ones for situation B. We set the current level of the convenience yield rate δ_t at 0.25 for both situations. When the spot price of this commodity moves up, the convenience yield rate tends to move up in the effect according to their correlation. Their correlation $\rho_{S\delta}$ is set at 0.75 for situation A and at 0.50 for situation B. These correlation parameters are estimated from WTI crude oil prices and convenience yield rates. Since we suppose this issuing firm sells this commodity to the market, the value of this issuer is positively correlated to the changes of this commodity prices and the convenience yields. Thus the commodity prices and the value of the issuer is set to behave with correlation $\rho_{SV} = 0.50$ for both situations. Also we set the correlation parameters $\rho_{V\delta}$ between the value of the issuer and the convenience yield at 0.50 for situation A and at 0.33 for situation B.

$F = 100$	$K = 20$	$\tau = 5$	$r = 0.04$	$S_t = 20$	$\sigma_s = 0.392$
$V_t = 200$	$\alpha_V = 0.02$	$\sigma_V = 0.3$	$\lambda_2 = -0.067$	$\delta_t = 0.25$	$\rho_{SV} = 0.5$
Situation A			Situation B		
$\kappa = 19$	$\mu_\delta = 0.32$	$\sigma_\delta = 1.31$	$\kappa = 4.5$	$\mu_\delta = 0.02$	$\sigma_\delta = 0.27$
$\lambda_3 = 0.214$	$\rho_{S\delta} = 0.75$	$\rho_{V\delta} = 0.5$	$\lambda_3 = 0.074$	$\rho_{S\delta} = 0.5$	$\rho_{V\delta} = 0.33$

Figure 2 shows that the graph of the commodity-linked bond prices as a function of the speed of adjustment κ. To calculate the bond prices for Figure 2 and Figure 3, we set $\mu_\delta = 0.1$, $\lambda_3 = 0.12$, $\sigma_\delta = 0.5$, $\rho_{S\delta} = 0.6$, and $\rho_{V\delta} = 0.4$ apart from situation A and B, we draw these two figures to see overall responses of bond prices and default probabilities to the values of κ. Figure 2 suggests that a smaller level of κ makes the bond prices higher than that of a larger level of κ. This means that the premium portion of bond prices decrease as κ become large, that is, movements of the convenience yield become more stable rather than that of smaller level of κ when other parameters are kept fixed. Figure 3 describes the default probabilities of the commodity-linked bond. This figure shows that the default probability become high as κ is at a smaller level. This result makes sense that the high premium, which means in part that the expected value of the payoff at the maturity is large, corresponds to the high default probability of this bond at the maturity.

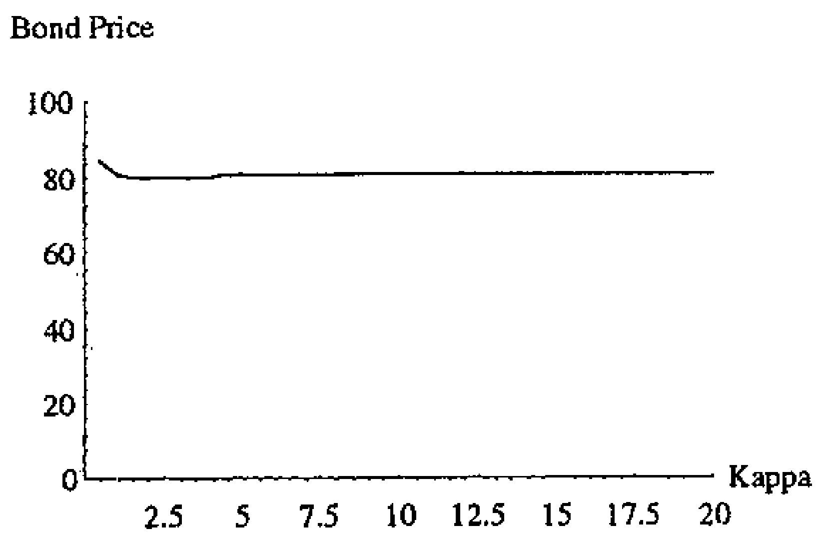


Figure 2.

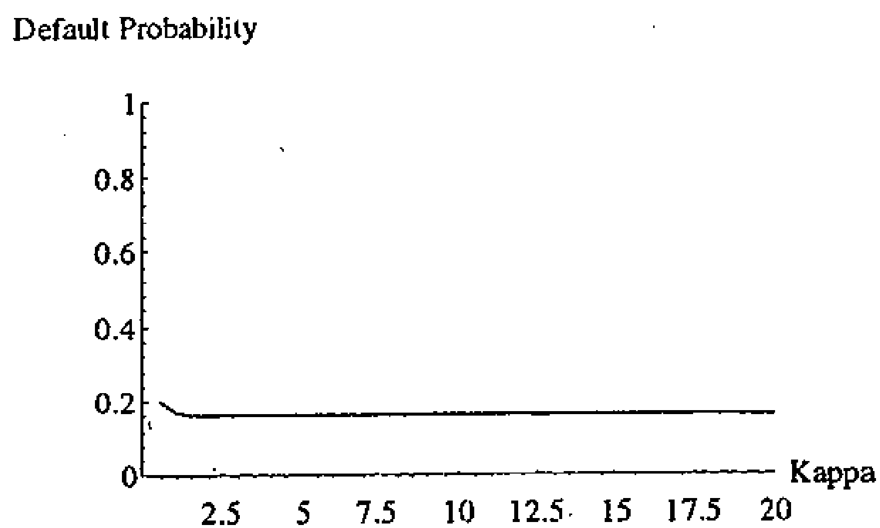


Figure 3.

Figure 4 suggests that the bond prices will increase as σ_δ increases for situation B, while the bond prices do not seem to be affected by the changes of σ_δ in situation A. This is because in a large level of κ, the convenience yield rate returns to its long term mean quickly even if σ_δ is at a large level. Consequently, the premium portion changes little as σ_δ become large. Figure 5 shows a graph of the default probabilities as a function of σ_δ.

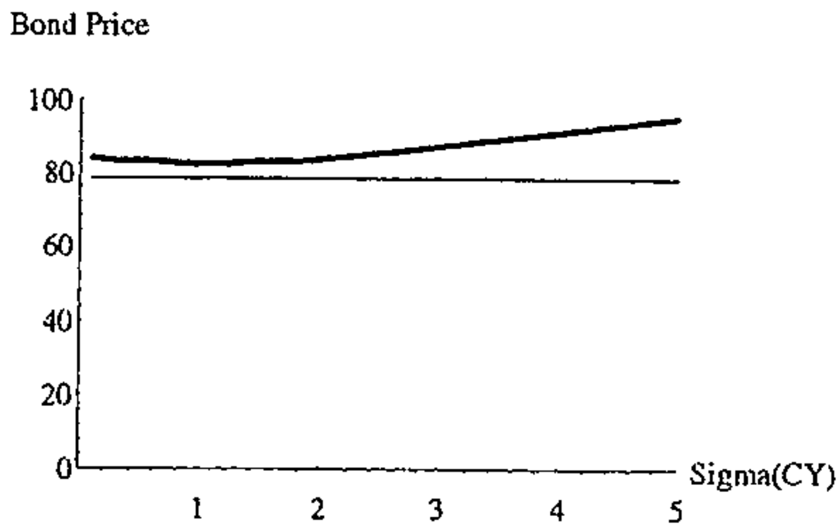


Figure 4. Thin line ———— : Situation A. Thick line ———— : Situation B.

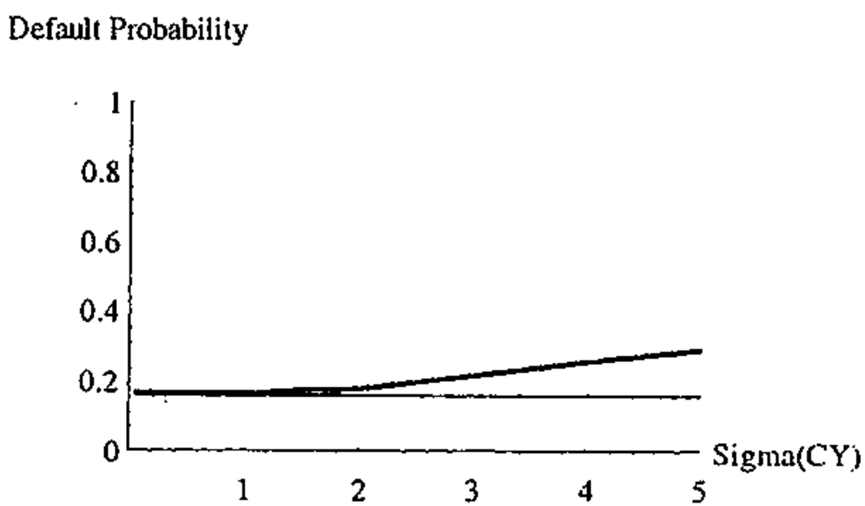


Figure 5. Thin line ———— : Situation A. Thick line ———— : Situation B.

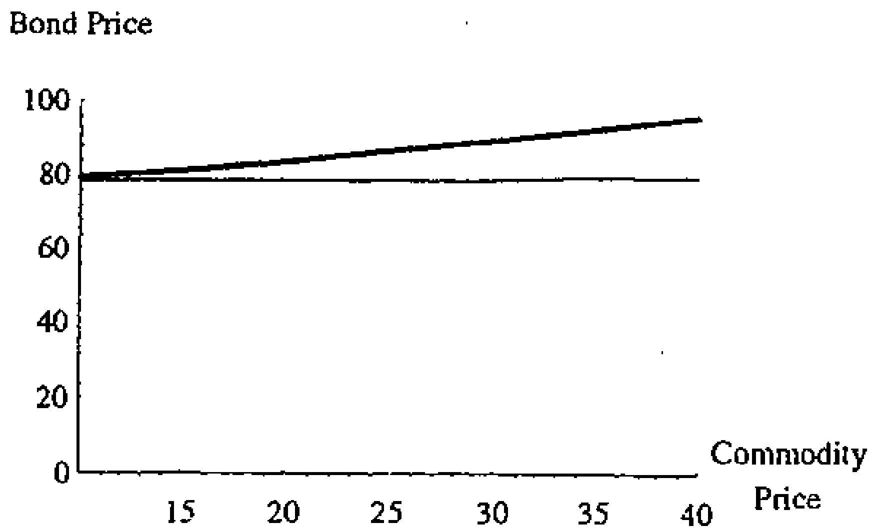


Figure 6. Thin line ———: Situation A. Thick line ———: Situation B.

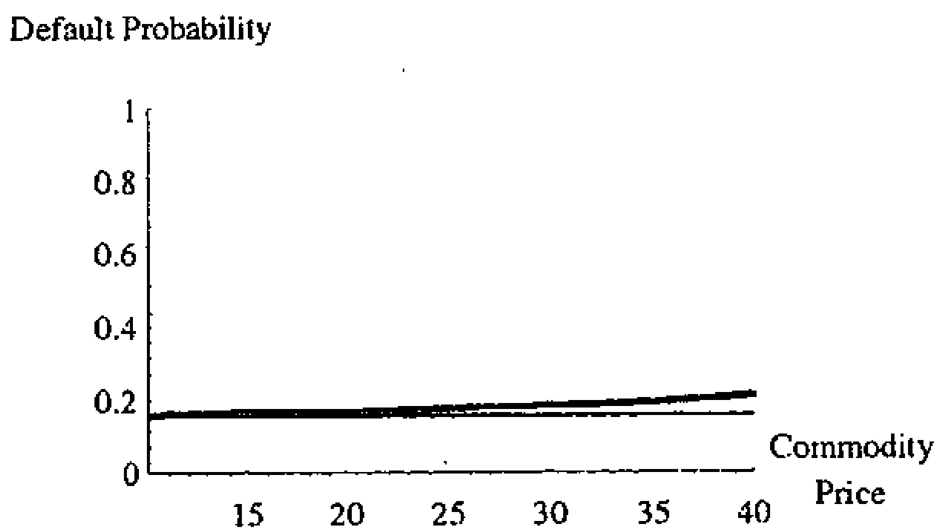


Figure 7. Thin line ———: Situation A. Thick line ———: Situation B.

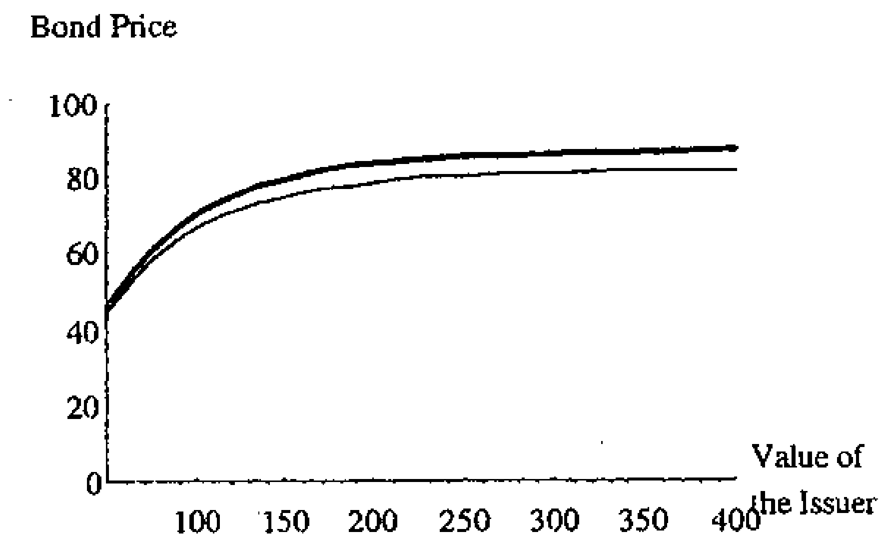


Figure 8. Thin line ——— : Situation A. Thick line ——— : Situation B.

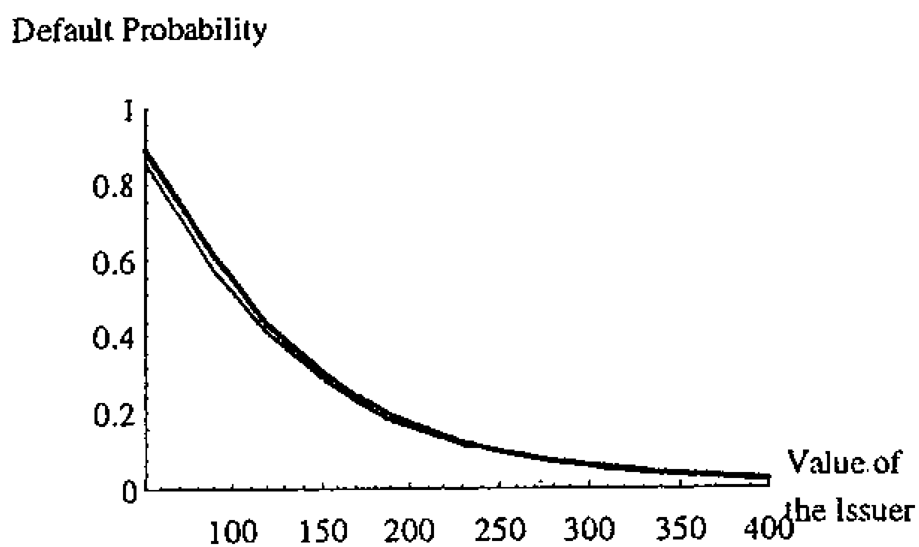


Figure 9. Thin line ——— : Situation A. Thick line ——— : Situation B.

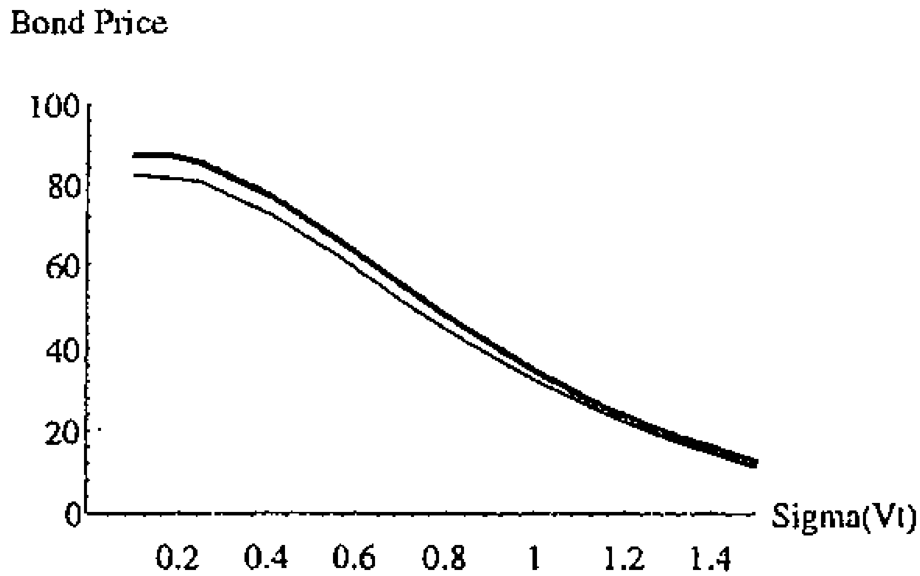


Figure 10. Thin line ______ : Situation A. Thick line ______ : Situation B.

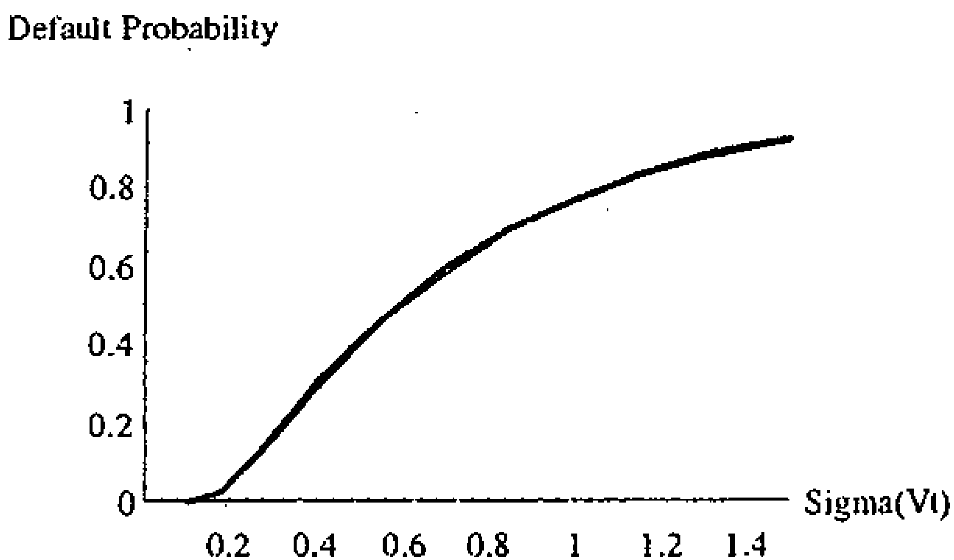


Figure 11. Thin line ______ : Situation A. Thick line ______ : Situation B.

Figure 6 shows that the higher the commodity prices S_t are, the more expensive the bond prices are. This is very natural. The strike price K is equal to 20, the premium increases as S_t moves across K from out of the money to in the money. Figure 7 is a graph of the default probabilities of the commodity-linked bond as a function of S_t. This figure also shows that the default probabilities become high as S_t becomes large which is the same as Figure 6.

Figure 8 through 11 are the graphs in relation to the value of the issuer. Figure 8 shows that the higher the value of the issuer is, the more expensive the bond price is. Figure 9 suggests that the default probability decreases as the value of the issuer V_t increases. This is very natural. If V_t is very small, the default probability of this bond at the maturity is anticipated to be high. As for Figure 10 and 11, we see that the larger the volatility of the value of the issuer σ_V is, the lower the bond price is and, at the same time, the default probability is high. These are also natural.

Appendix.

Taken from Miura&Yamauchi (1998).

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Appendix: On Convenience Yields

Some explanation

1.2. CONVENIENCE YIELD

The owner of the commodity has the rights (this is an option) to decide how he/she will treat the commodity; sell, lend, or store it, or even consume it. As for the consumption-use commodities such as crude oil or copper, the owner may consume it for his/her own manufacturing activities, or he/she may also store it for his/her future consumption or future sell-out. The owner of the futures contracts or the other contingent claims, however, does not have this rights because of lack of storage until the maturity.

storage until the maturity.

The commodities prices are also seen to change with regards to the storage level of the participants in the market. Since all the participants make their own decision taking account of their own current and future perspective of inventory levels and time intervals, the market prices will change as aggregated results of each activity conducted by them. As Duffie (1989) discusses, the convenience yield is seen as the value of the option to sell out of storage. We will thus assume that the yield will change in relation to the scarcity of the commodity in the market. A low inventory level in the market, that is, scarcity of storage, leads to be backwardation of a market where the futures prices of distant contract months are lower than those of the nearby (see, for example, Edwards and Ma, 1992). This means that backwardation occurs when there is a shortage of the available physical commodity. This shortage implies the following attitude of the holder; 'the holder of the physical commodity are unwilling to part with it, even for short period of time (Edwards and Ma, 1992)' and thus generates the convenience yields. In this respect, we may assume that there is an inverse relationship between the changes of the convenience yield and the changes of current inventory level in the market. Kaldor (1939) and Working (1948) examined and affirmed this hypothesis.[2]

A statistical analysis for the net marginal convenience yield can be done using spot and futures prices. Brennan (1991) squeezed out the net marginal convenience yield from futures prices of gold, silver, platinum, copper, No. 2 heating oil, lumber, and plywood. By analyzing those data, he showed their mean-reverting movements. Gibson and Schwartz (1990, 1993) used the relations between futures prices with different contract months to estimate parameter values in the models for the net marginal convenience yields' movements and utilized the estimated values for their numerical pricing of the contingent claims.

Appendix A: Proof for the Independence of λ_1, λ_2, and λ_3 on the Choice of the Assets

By the standard no-arbitrage argument, we derived constants λ_1, λ_2, and λ_3 for the choice of the three assets, namely, 1st, 2nd, and 3rd bonds in Section 2. They are tentatively dependent on the choice of three assets. In this appendix, we prove its independence on the choice of the assets. First, we exchange 3rd bonds to 4th bonds. The same no-arbitrage argument is valid for this new portfolio and we obtain a new set of constants λ'_1, λ'_2, and λ'_3 with the following equations:

$$\begin{bmatrix} \varphi_{B,1} - r \\ \varphi_{B,2} - r \\ \varphi_{B,4} - r \\ \alpha_S + \delta_t - r \end{bmatrix} = \lambda'_1 \cdot \begin{bmatrix} \pi_S^1 \\ \pi_S^2 \\ \pi_S^4 \\ \sigma_S \end{bmatrix} + \lambda'_2 \cdot \begin{bmatrix} \pi_V^1 \\ \pi_V^2 \\ \pi_V^4 \\ 0 \end{bmatrix} + \lambda'_3 \cdot \begin{bmatrix} \pi_\delta^1 \\ \pi_\delta^2 \\ \pi_\delta^4 \\ 0 \end{bmatrix} . \tag{A.1}$$

From equations $\mathbf{D} = \lambda_1 \cdot \mathbf{A} + \lambda_2 \cdot \mathbf{B} + \lambda_3 \cdot \mathbf{C}$ and (A.1) and (5) (the assumption of non-singularity), we obtain

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda'_1 \\ \lambda'_2 \\ \lambda'_3 \end{bmatrix} = \begin{bmatrix} \pi_S^1 & \pi_V^1 & \pi_\delta^1 \\ \pi_S^2 & \pi_V^2 & \pi_\delta^2 \\ \sigma_S & 0 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \varphi_{B,1} - r \\ \varphi_{B,2} - r \\ \alpha_S + \delta_t - r \end{bmatrix} .$$

This procedure can be iterated. The iteration is done for exchanging 2nd bond to 5th bond, and 1st bond to 6th bond. Then we have a unique time dependent constant set, λ_1, λ_2, and λ_3. Thus, these constants are independent of the choice of assets. Each of these is called the market price of risk.

Of course, we can imply the same result to the set of constants λ_1, λ_2, λ_3, and λ_4 in section 4.2 and also λ_1^*, λ_2^*, and λ_3^* in Appendix C.

Appendix B: Decomposition of $\mathbf{G} \cdot \mathbf{G}^{\mathrm{T}}$

$\mathbf{G}$ is a matrix such that

$$\mathbf{G} \cdot \mathbf{G}^{\mathrm{T}} = \begin{bmatrix} \tilde{S}_t^2 \sigma_S^2 & \tilde{S}_t \tilde{V}_t \sigma_S \sigma_V \rho_{SV} & \tilde{S}_t \sigma_S \sigma_\delta \rho_{S\delta} \\ \tilde{S}_t \tilde{V}_t \sigma_S \sigma_V \rho_{SV} & \tilde{V}_t^2 \sigma_V^2 & \tilde{V}_t \sigma_V \sigma_\delta \rho_{V\delta} \\ \tilde{S}_t \sigma_S \sigma_\delta \rho_{S\delta} & \tilde{V}_t \sigma_V \sigma_\delta \rho_{V\delta} & \sigma_\delta^2 \end{bmatrix}.$$

To get $\mathbf{G}$, the following decomposition helps:

$$\mathbf{G} \cdot \mathbf{G}^{\mathrm{T}} = \begin{bmatrix} \tilde{S}_t \sigma_S & 0 & 0 \\ 0 & \tilde{V}_t \sigma_V & 0 \\ 0 & 0 & \sigma_\delta \end{bmatrix} \cdot \begin{bmatrix} 1 & \rho_{SV} & \rho_{S\delta} \\ \rho_{SV} & 1 & \rho_{V\delta} \\ \rho_{S\delta} & \rho_{V\delta} & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{S}_t \sigma_S & 0 & 0 \\ 0 & \tilde{V}_t \sigma_V & 0 \\ 0 & 0 & \sigma_\delta \end{bmatrix}.$$

Next, we decompose the second matrix of the RHS above

$$\begin{bmatrix} 1 & \rho_{SV} & \rho_{S\delta} \\ \rho_{SV} & 1 & \rho_{V\delta} \\ \rho_{S\delta} & \rho_{V\delta} & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & \bar{e} & f \end{bmatrix} \cdot \begin{bmatrix} a & b & d \\ 0 & c & \bar{e} \\ 0 & 0 & f \end{bmatrix}^{\mathrm{T}},$$

where

$$\begin{cases} a^2 = 1, b^2 + c^2 = 1, d^2 + \bar{e}^2 + f^2 = 1 \\ ab = \rho_{SV}, ad = \rho_{S\delta}, bd + c\bar{e} = \rho_{V\delta} \end{cases}.$$

When we assign $a = 1$, we have

$$b = \rho_{SV}; d = \rho_{S\delta} \text{ and } c^2 = 1 - \rho_{SV}^2 .$$

Now we suppose c and f be positive (of course, negative values are feasible. But we select positive values for c and f as one of the choices),

$$c = \sqrt{1 - \rho_{SV}^2}, \bar{e} = \frac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\sqrt{1 - \rho_{SV}^{2'}}}, f = \sqrt{1 - \frac{(\rho_{V\delta} - \rho_{SV}\rho_{S\delta})^2}{1 - \rho_{SV}^2} - \rho_{S\delta}^2}.$$

Then we can write **G** as follows:

$$
\mathbf{G} = \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 \\ 0 & \tilde{V}_t\sigma_V & 0 \\ 0 & 0 & \sigma_\delta \end{bmatrix} \cdot \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & \bar{e} & f \end{bmatrix}
$$

$$
= \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 \\ \tilde{V}_t\sigma_V\rho_{SV} & \tilde{V}_t\sigma_V \cdot c & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot \bar{e} & \sigma_\delta \cdot f \end{bmatrix}.
$$

This argument can be used to obtain a choice of **G** for the SDE (18) in relation to the PDE (17) in subsection 4.3 and the same thing applies to the SDE (C.5) in Appendix C.

Appendix. Subsection 4.2 of the paper

Derivation of the PDE

The case where interest rate is not constant.

But it follows Ornstein=Uhlenbech process.

Very much the same argument as the case of constant interest rate as long as mathematical argument concerns.

4. Closed Pricing Formula for the Commodity-Linked Bonds $B(S_t, V_t, \delta_t, r_t, \tau)$

In this section, we describe a straight extension of Section 2 where the instantaneous interest rate changes stochastically, following another Ornstein–Uhlenbech process.

4.1. ASSUMPTIONS FOR THE PRICING FORMULA OF COMMODITY-LINKED BONDS $B(S_t, V_t, \delta_t, r_t, \tau)$

Assume the same situation as we postulated in the Section 2 except that the interest rate behaves stochastically

$$\mathrm{d}r_t = g(\mu_r - r_t)\mathrm{d}t + \sigma_r \cdot \mathrm{d}W_r \,, \tag{14}$$

where W_r is another standard Wiener process and correlations are such that

$$\mathrm{d}W_s \cdot \mathrm{d}W_r = \rho_{Sr}\mathrm{d}t, \ \mathrm{d}W_V \cdot \mathrm{d}W_r = \rho_{Vr}\mathrm{d}t, \ \text{ and } \ \mathrm{d}W_\delta \cdot \mathrm{d}W_r = \rho_{\delta r}\mathrm{d}t \,.$$

4.2. PARTIAL DIFFERENTIAL EQUATION

In this subsection, we derive the PDE for the pricing function of the commodity-linked bond $B_i(S_t, V_t, \delta_t, r_t, \tau)$. By using Ito's lemma, we obtain the following equation for the i-th commodity-linked bond ($i = 1, 2, \cdots, N$);

$$\frac{\mathrm{d}B_i}{B_i} = \Psi_{B,i} \cdot \mathrm{d}t + \eta_S^i \cdot \mathrm{d}W_S + \eta_V^i \cdot \mathrm{d}W_V + \eta_\delta^i \cdot \mathrm{d}W_\delta + \eta_r^i \cdot \mathrm{d}W_r , \tag{16}$$

where

$$\Psi_{B,i} = \frac{\left\{\begin{array}{l} \frac{\partial B_i}{\partial S} S\alpha_S + \frac{\partial B_i}{\partial V} V\alpha_V + \frac{\partial B_i}{\partial \delta}\kappa(\mu_\delta - \delta) + \frac{\partial B_i}{\partial r} g(\mu_r - r) - \frac{\partial B_i}{\partial \tau} \\ +\frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial S^2} S^2\sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial V^2} V^2\sigma_V^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial \delta^2}\sigma_\delta^2 + \frac{1}{2} \cdot \frac{\partial^2 B_i}{\partial r^2}\sigma_r^2 \\ +\frac{\partial^2 B_i}{\partial S \partial V} SV\sigma_S\sigma_V\rho_{SV} + \frac{\partial^2 B_i}{\partial S \partial \delta} S\sigma_S\sigma_\delta\rho_{S\delta} + \frac{\partial^2 B_i}{\partial S \partial r} S\sigma_S\sigma_r\rho_{Sr} \\ +\frac{\partial^2 B_i}{\partial V \partial \delta} V\sigma_V\sigma_\delta\rho_{V\delta} + \frac{\partial^2 B_i}{\partial V \partial r} V\sigma_V\sigma_r\rho_{Vr} + \frac{\partial^2 B_i}{\partial \delta \partial r}\sigma_\delta\sigma_r\rho_{\delta r} \end{array}\right\}}{B_i},$$

$$\eta_S^i = \frac{\partial B_i}{\partial S} \cdot \frac{S\sigma_S}{B_i}, \quad \eta_V^i = \frac{\partial B_i}{\partial V} \cdot \frac{V\sigma_V}{B_i},$$

$$\eta_\delta^i = \frac{\partial B_i}{\partial \delta} \cdot \frac{\sigma_\delta}{B_i}, \quad \eta_r^i = \frac{\partial B_i}{\partial r} \cdot \frac{\sigma_r}{B_i}.$$

With the same standard no-arbitrage argument used in Section 2, we obtain

$$\begin{aligned}
&\frac{\partial B}{\partial S} S_t(r_t - \delta_t) + \frac{\partial B}{\partial V} V_t(\alpha_V - \lambda_2 \sigma_V) + \frac{\partial B}{\partial \delta}\{\kappa(\mu_\delta - \delta_t) - \lambda_3 \sigma_\delta\} + \\
&+ \frac{\partial B}{\partial r}\{g(\mu_r - r_t) - \lambda_4 \sigma_r\} + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial S^2} S_t^2 \sigma_S^2 + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial V^2} V_t^2 \sigma_V^2 + \\
&+ \frac{1}{2} \cdot \frac{\partial^2 B}{\partial \delta^2} \sigma_\delta^2 + \frac{1}{2} \cdot \frac{\partial^2 B}{\partial r^2} \sigma_r^2 + \frac{\partial^2 B}{\partial S \partial V} S_t V_t \sigma_S \sigma_V \rho_{SV} + \\
&+ \frac{\partial^2 B}{\partial S \partial \delta} S_t \sigma_S \sigma_\delta \rho_{S\delta} + \frac{\partial^2 B}{\partial S \partial r} S_t \sigma_s \sigma_r \rho_{Sr} + \frac{\partial^2 B}{\partial V \partial \delta} V_t \sigma_V \sigma_\delta \rho_{V\delta} + \\
&+ \frac{\partial^2 B}{\partial V \partial r} V_t \sigma_V \sigma_r \rho_{Vr} + \frac{\partial^2 B}{\partial \delta \partial r} \sigma_\delta \sigma_r \rho_{\delta r} - \frac{\partial B}{\partial \tau} - r_t \cdot B = 0\,,
\end{aligned} \tag{17}$$

where λ_2, λ_3, and λ_4 are the market prices of risk for the value of the issuer, the net marginal convenience yield, and the interest rate, respectively. Equation (17) is the PDE which every $B(S_t,\ V_t,\ \delta_t,\ r_t,\ \tau)$ must satisfy.

Appendix

Subsection 4.3.

Detailed derivation of joint distribution.

4.3. CLOSED PRICING FORMULA

In this subsection, we derive the closed pricing formula of the commodity-linked bond $B(S_t, V_t, \delta_t, r_t, \tau)$ by applying the Feynman–Kac Theorem. In the calculation of the expected value of the payoff function (4), we need the joint distribution function of $[\tilde{S}_t, \tilde{V}_t, \tilde{\delta}_t, \tilde{r}_t]$ based on PDE (17). From (17), we have the following SDE:

$$d\begin{bmatrix} \tilde{S}_t \\ \tilde{V}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix} = \begin{bmatrix} \tilde{S}_t(\tilde{r}_t - \tilde{\delta}_t) \\ \tilde{V}_t(\alpha_V - \lambda_2\sigma_V) \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \\ g(\mu_r - \tilde{r}_t) - \lambda_4\sigma_r \end{bmatrix} \mathrm{d}t + \begin{bmatrix} \tilde{S}_t\sigma_S & 0 & 0 & 0 \\ \tilde{V}_t\sigma_V\rho_{SV} & \tilde{V}_t\sigma_V \cdot \bar{e} & 0 & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot f & \sigma_\delta \cdot h & 0 \\ \sigma_r\rho_{Sr} & \sigma_r \cdot \bar{g} & \sigma_r \cdot i & \sigma_r \cdot j \end{bmatrix} \cdot \mathrm{d}\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \\ \tilde{Z}_{4,t} \end{bmatrix}, \tag{18}$$

where

$$\begin{cases} \bar{e} = \sqrt{1-\rho_{SV}^2}, \quad f = \dfrac{\rho_{V\delta} - \rho_{SV}\rho_{S\delta}}{\bar{e}}, \quad \bar{g} = \dfrac{\rho_{Vr} - \rho_{SV}\rho_{Sr}}{\bar{e}} \\ h = \sqrt{1 - p_{S\delta}^2 - f^2}, \quad i = \dfrac{\rho_{\delta r} - \rho_{S\delta}\rho_{Sr} - f \cdot \bar{g}}{h}, \quad j = \sqrt{1 - \rho_{Sr}^2 - \bar{g}^2 - i^2} \end{cases}$$

and $\tilde{Z}_{1,t}$, $\tilde{Z}_{2,t}$, $\tilde{Z}_{3,t}$, $\tilde{Z}_{4,t}$, are independent standard Wiener processes.

Next, to transform the *non-linear* SDE (18) to *linear one*, let $\tilde{P}_t = \log\ \tilde{S}_t$ and $\tilde{J}_t = \log\ \tilde{V}_t$. Then we have the following SDEs for $\tilde{P}_t$ and $\tilde{J}_t$ by applying Ito's lemma.

$$d\begin{bmatrix} \tilde{\rho}_t \\ \tilde{\tau}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix} = \begin{bmatrix} -\tilde{\delta}_t + r_t - \frac{1}{2}\sigma_S^2 \\ \alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2 \\ \kappa(\mu_\delta - \tilde{\delta}_t) - \lambda_3\sigma_\delta \\ g(\mu_r - \tilde{r}_t) - \lambda_4\sigma_r \end{bmatrix} \mathrm{d}t + \\ + \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ \sigma_V\rho_{SV} & \sigma_V \cdot \bar{e} & 0 & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot f & \sigma_\delta \cdot h & 0 \\ \sigma_r\rho_{Sr} & \sigma_r \cdot \bar{g} & \sigma_r \cdot i & \sigma_r \cdot j \end{bmatrix} \cdot \mathrm{d}\begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \\ \tilde{Z}_{4,t} \end{bmatrix}. \tag{19}$$

Here, we need to solve the linear SDE.

Solution form is given in a text book(Arnold's,for example).

Or, you can just differentiate (stochastically) the solution,

then you will see that it satisfies the SDE.

By solving the stochastic differential Equations (19), we derive the joint probability density function of $[\tilde{P}_t, \tilde{J}_t, \tilde{\delta}_t, \tilde{r}_t]$ for the time interval $[t_0, T]$ using theorem 8.2.2 in Arnold (1973, p. 129). By theorem 8.2.2, the SDE

$$d\tilde{\mathbf{Q}}_t = (\mathbf{A}(t) \cdot \tilde{\mathbf{Q}}_t + \mathbf{a}(t))dt + \mathbf{B}(t) \cdot d\tilde{\mathbf{Z}}_t$$

has the solution

$$\tilde{\mathbf{Q}}_t = \Phi_t(\mathbf{Q}_{t_0} + \int_{t_0}^{t} \Phi_s^{-1} \cdot \mathbf{a}(s) \cdot ds + \int_{t_0}^{t} \Phi_s^{-1} \cdot \mathbf{B}(s) \cdot d\tilde{\mathbf{Z}}_s) \tag{20}$$

with the initial value $\mathbf{Q}_{t_0}$, where with the initial value $\mathbf{Q}_{t0}$, where

$$\tilde{\mathbf{Q}}_t = \begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix}, \quad \mathbf{Q}_{t0} = \begin{bmatrix} P_{t0} \\ J_{t0} \\ \delta_{t0} \\ r_{t0} \end{bmatrix},$$

$$\mathbf{A}(t) = \mathbf{A} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & -g \end{bmatrix},$$

$$\mathbf{a}(t) = \mathbf{a} = \begin{bmatrix} -\frac{1}{2}\sigma_S^2 \\ \alpha_V - \lambda_2\sigma_V - \frac{1}{2}\sigma_V^2 \\ \kappa\mu_\delta - \lambda_3\sigma_\delta \\ g\mu_r - \lambda_4\sigma_r \end{bmatrix},$$

$$\mathbf{B(t)} = \mathbf{B} = \begin{bmatrix} \sigma_S & 0 & 0 & 0 \\ \sigma_V\rho_{SV} & \sigma_V \cdot \bar{e} & 0 & 0 \\ \sigma_\delta\rho_{S\delta} & \sigma_\delta \cdot f & \sigma_\delta \cdot h & 0 \\ \sigma_r\rho_{Sr} & \sigma_r \cdot \bar{g} & \sigma_r \cdot i & \sigma_r \cdot j \end{bmatrix}, \quad \text{and } \tilde{\mathbf{Z}}_t = \begin{bmatrix} \tilde{Z}_{1,t} \\ \tilde{Z}_{2,t} \\ \tilde{Z}_{3,t} \\ \tilde{Z}_{4,t} \end{bmatrix}.$$

In this solution (20), Φ_t stands for the fundamental matrix of $\dot{\tilde{\mathbf{Q}}}_t = \mathbf{A}(t) \cdot \tilde{\mathbf{Q}}_t \cdot \Phi_t$ and its inverse are given by

$$\Phi_t = e^{\mathbf{A}(t-t_0)} = \begin{bmatrix} 1 & 0 & \frac{1}{\kappa}(e^{-\kappa(t-t_0)} - 1) & \frac{1}{g}(1 - e^{-g(t-t_0)}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-\kappa(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{-g(t-t_0)} \end{bmatrix}, \tag{21}$$

$$\Phi_t^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{\kappa}(e^{\kappa(t-t_0)} - 1) & \frac{1}{g}(1 - e^{g(t-t_0)}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\kappa(t-t_0)} & 0 \\ 0 & 0 & 0 & e^{g(t-t_0)} \end{bmatrix}, \tag{22}$$

By substituting (21) and (22) into (20), we have

$$\begin{bmatrix} \tilde{P}_t \\ \tilde{J}_t \\ \tilde{\delta}_t \\ \tilde{r}_t \end{bmatrix} = \begin{bmatrix} \alpha(\pi) \\ \beta(\pi) \\ \gamma(\pi) \\ \varepsilon(\pi) \end{bmatrix} + \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t \\ \tilde{z}_t \\ \tilde{w}_t \end{bmatrix},$$

For further details, please see the paper.

After some calculus, we arrive at the following.

$f(x_t, y_t, r_t^*)$ is the joint density function such that

$$f(x_t, y_t, r_t^*) = \frac{1}{(2\pi)^{3/2} \cdot \sqrt{\det \sum}} \cdot \exp\left\{-\frac{1}{2}\mathbf{v}^{\mathrm{T}} \sum^{-1} \mathbf{v}\right\},$$

where $\mathbf{v} = \begin{bmatrix} x_t \\ y_t \\ r_t^* \end{bmatrix}$ and $\sum$ is its variance-covariance matrix which is given by

$$\sum = \begin{bmatrix} Var(x_t) & \sigma_{xy} & \sigma_{xr^*} \\ \sigma_{xy} & Var(y_t) & \sigma_{yr^*} \\ \sigma_{xr^*} & \sigma_{yr^*} & Var(r_t^*) \end{bmatrix},$$

where