2nd. Day. April 27. (3+2 =5hours)

[Exotic Options.] (European Type)

: (i).Simple Exotics as an introduction.
Pay-Later. Chooser, Supershare, etc.
Parisian ,Asian,

: (ii).Edokko Options.

: (iii).Stochastic Corridors

: (iv) Weather Derivatives

Section 1. Some Exotic Derivatives, And a brief review of Black-Scholes option pricing theory

Pay-off functions identify the derivatives.

: simple exotics with modified pay-off functions.

And some more complicated exotics.

: Outline of Black-Scholes option pricing framework;

PDE approach and Martingale approach.

: Linearity of Pay-off function ⇔ Linearity of pricing functions

Brownian quantile is not Markov?

Description (ideas and definitions) of simple exotics.
(These will be done on the board .)(no slides; sorry).

: simple exotics with modified pay-off functions.

Pay-later and Supershare.

: Chooser.

:Max(or Min) of two.

:Asian.

: Parisian.

: Edokko (application of alpha-quantiles)

PDE Approach.

$$\frac{\partial C}{\partial t}+\frac{\partial C}{\partial S_t}rS_t+\frac{1}{2}\frac{\partial^2 C}{\partial S_t{}^2}\sigma^2 S_t{}^2-rC=0.$$

This is the partial diferential equation which the price functions of all the securities in this market have to satisfy.

A boundary condition $\mathrm{C}(\mathrm{T}, \mathrm{S}_T)=h(\mathrm{S}_T)$,

which is the pay-off of the derivative,

Martingael Approach.

Now, $\mathrm{V}_T=B_T A_T=B_T(B_T{}^{-1}X)=X$ and $\{\mathrm{V}_t, t\in[0,T]\}$ being self-financing imply that under no-arbitrage assumption, V_t copys the price of option with pay-off X and that V_0 must coincide with the price of the option at time zero.

$$\mathrm{V}_0=B_0A_0=A_0=E_Q[B_T{}^{-1}X\,|\,F_0]=E_Q[B_T{}^{-1}\max\{\mathrm{S}_T\text{-K},0\}\,|\,F_0]$$

$$=\mathrm{e}^{-rT}E_Q[\max\{\mathrm{S}_0\mathrm{e}^{(r-\frac{1}{2}\sigma^2)T+\sigma\widetilde{W_T}}\text{-K},0\}]$$

: Brief descriptions of Black-Scholes option pricing framework.
(1). PDE Approach.

Contracts in practice (over-the-counter or exchange-traded) and theoretical assumptions.

: underlying assets (or variables)(objectively observable)

<= assume a stochastic model.

: Time interval [0,T].

: pay-off <= a function of the underlying variable(s).

: market conditions.

<= constant interest rate. Infinitesimal trading.

<= no- arbitrage assumption. And Completeness of market.

Memo

: give brief description of both approaches for pricing, and explain importance of Pay-off function and self-financing property.

: remark on linear homogeneity of pay-off function.

: show that average option can be treated by PDF (taking from Wilmott's , or Shreve's(?) book.)

$S_t : t \in [0,\infty).$

Geometric Brownian Motion process.

$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$ (or, $dS_t = S_t(\mu dt + \sigma dW_t)$),

where W_t is a standard Brownian process.

: r be a constant interest rate.

: [0,T] . option contract matures at time T.

: Pay-off $\mathrm{h}(\mathrm{S}_T)$.

e.g. Call option: $\mathrm{h}(\mathrm{S}_T) = \max\{\mathrm{S}_T - K, 0\}$. K be determined in the contract.

: Let a price of an option be a function C(t,S_t) of time t and the underlying variable S_t.

It also depends on K and market conditions: r and σ^2, but these are constants (not time-varying).

: look at the stochastice movement of C(t,S_t) by Ito-calculus.

$C(t+dt, S_{t+dt}) - C(t, S_t),\ dt \to 0.$

Taylor expansion and Ito-calculus scheme.

$$dC = \frac{\partial C}{\partial t}\mathrm{dt} + \frac{\partial C}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t{}^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 C}{\partial t^2}(dt)^2 + \frac{\partial^2 C}{\partial S_t \partial t}(dS_t)(dt) + ...$$

$$= \frac{\partial C}{\partial t}\mathrm{dt} + \frac{\partial C}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t{}^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 C}{\partial t^2}(dt)^2$$

$$=(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t{}^2}\sigma^2 S_t{}^2)\mathrm{dt} + \frac{\partial C}{\partial S_t}\sigma S_t dW_t$$

Let

$$\mu_C = \frac{1}{C}(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t{}^2}\sigma^2 S_t{}^2), \text{ and } \sigma_C = \frac{1}{C}(\frac{\partial C}{\partial S_t}\sigma S_t)$$

Then, we write; $\frac{dC}{C} = \mu_C dt + \sigma_C dW_t.$

We can write in the same way for any other price functions F,G,

of financial instruments(derivatives) based on the same S.

Let us make a "riskless" portfolio at a time point $t \in [0,T]$.

Take two arbitrary assets with price functions C and F.

Invest an amount of money Π on these assets.

Let w_C and w_F be investment ratio with $w_C + w_F = 1$.

$$\Pi = w_C \Pi + w_F \Pi = \frac{w_C \Pi}{C} C + \frac{w_F \Pi}{F} F$$

(please notice that something is implicitly assumed in the following intuitive argument.)

Then,

$$\frac{d\Pi_t}{\Pi_t} = \left(\frac{\Pi_{t+dt} - \Pi_t}{\Pi_t}, dt \to 0\right) = \frac{1}{\Pi_t}\left(\frac{w_C \Pi_t}{C} dC + \frac{w_F \Pi_t}{F} dF\right) = w_C \frac{dC}{C} + w_F \frac{dF}{F}$$

(self-financing property of the portfolio is assumed here.)

$\frac{dC}{C}$ and $\frac{dF}{F}$ were given under Ito-calculus. Substitute them from above.

Then, we have; $\frac{d\Pi_t}{\Pi_t} = (w_C \mu_C + w_F \mu_F) dt + (w_C \sigma_C + \sigma_F w_F) dW_t$

The portfolio Π^*: (w^*_C, w^*_F) that makes $(w_C \sigma_C + \sigma_F w_F) = 0$ under $(w_C + w_F) = 1$ be selected;

$$w^*_C = \frac{\sigma_F}{\sigma_F - \sigma_C} \quad \text{and} \quad w^*_F = \frac{-\sigma_C}{\sigma_F - \sigma_C}$$

$\frac{d\Pi^*_t}{\Pi^*_t} = (w^*_C \mu_C + w^*_F \mu_F) dt$: a "riskless" portfolio at a time point $t \in [0,T]$.

No-Arbitrage arguement.

$$\frac{d\Pi^*_t}{\Pi^*_t} = (\mathrm{w}^*_C \mu_C + \mathrm{w}^*_F \mu_F)dt, \text{ and } \frac{d\Pi^*_t}{\Pi^*_t} = rdt$$

Under no arbitage assumption, $\mathrm{w}^*_C \mu_C + \mathrm{w}^*_F \mu_F = r$.

Plug in w^*_C and w^*_F to obtain;

$$\sigma_F \mu_C - \sigma_C \mu_F = r(\sigma_F - \sigma_C).$$

Thus,

$$\frac{\mu_C - r}{\sigma_C} = \frac{\mu_F - r}{\sigma_F}.$$

Note that C and F were chosen arbitrarily.

Hence this quantity is common to all the securities in this market.

Let this quantity be denoted by λ.

From the above arguement, λ can be dependent on time t, but for the underlying asset, its price S_t has $\mu_S = \mu$ and $\sigma_S = \sigma$ which are constants. Therefore, the common quantity $\lambda = \frac{\mu - r}{\sigma}$ is a constant in this market.

Now put the definition of μ_C and σ_C into $\dfrac{\mu_C - r}{\sigma_C} = \lambda$.

Then, we have;

$$\frac{\frac{1}{C}\left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^{\ 2}}\sigma^2 S_t^{\ 2}\right)\text{-r}}{\frac{1}{C}\left(\frac{\partial C}{\partial S_t}\sigma S_t\right)} = \lambda$$

that is;

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t}(\mu - \lambda\sigma)S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^{\ 2}}\sigma^2 S_t^{\ 2} - rC = 0.$$

Recall that $\mu - \lambda\sigma = r,$ then we finally have;

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t} r S_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^{\ 2}}\sigma^2 S_t^{\ 2} - rC = 0.$$

This is the partial diferential equation which the price functions of all the securities in this market have to satisfy.

A boundary condition $\mathrm{C(T, S}_T) = h(\mathrm{S}_T)$,

which is the pay-off of the derivative,

will identify what derivative we are working on.

In order to solve this PDE for C,

we use Feynman-Kac Theorem for the current case

of a Brownian motion with drift r ; it states

$$\mathrm{C}(\mathrm{t}, \mathrm{S}_t) = E[h(X_T)e^{-r(T-t)} \mid X_t = S_t] ,$$

where $\frac{\mathrm{dX}_t}{\mathrm{X}_t} = rdt + \sigma dV_t$, and V_t is another zero-drift Brownian motion,

satisfies;

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t} rS_t + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^{\ 2}}\sigma^2 S_t^{\ 2} - rC = 0$$

with a boundary condition $\mathrm{C}(\mathrm{T},\mathrm{S}_T) = h(S_T)$.

: Brief descriptions of Black-Scholes option pricing framework.
Martingale Approach.

S_t, r, [0,T].

Let $B_t = e^{rt}$: Money market account.

(Ω, F, P) and $\{\mathrm{F}_t\}$

$$S_t = S_0 e^{\mu t + \sigma W_t} \quad (\text{or, } dS_t = S_t(\mu + \frac{1}{2}\sigma^2)dt + \sigma S_t dW_t))$$

: Pay-off $h(S_T)$.

Put $Z_t = (B_t)^{-1} S_t = S_0 e^{(\mu - r)t + \sigma W_t}$.

By Ito's stochastic calculus,

$$dZ_t = Z_t\{(\mu - r)dt + \sigma dW_t + \frac{1}{2}\sigma^2 dt\} = Z_t\{(\mu - r + \frac{1}{2}\sigma^2)dt + \sigma dW_t\}$$

$$= Z_t \sigma d\left(W_t + \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma} t \right).$$

Let $\lambda = \dfrac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}$ and define Q by $\dfrac{dQ}{dP} = e^{\lambda W_T - \frac{1}{2}\lambda^2 T}$.

From C-M-G-Ma theorem, $W_t = W_t + \lambda t$ is a standard Brownian motion under Q.

So, $\dfrac{dZ_t}{Z_t} = \sigma dW_t$ (or, $Z_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$) and $\mathrm{E}_Q[\dfrac{dZ_t}{Z_t}] = 0.$

Now, note that Z_t is a martingale under Q ;

for $0 \le u < t < T$,

$$\mathrm{E}_Q[Z_t \mid F_u] = \mathrm{E}_Q[e^{-\frac{1}{2}\sigma^2 u + \sigma \tilde{W}_u} e^{-\frac{1}{2}\sigma^2 (t-u) + \sigma (\tilde{W}_t - \tilde{W}_u)} \mid F_u]$$

$$= Z_u \mathrm{E}_Q[e^{-\frac{1}{2}\sigma^2 (t-u) + \sigma (\tilde{W}_t - \tilde{W}_u)} \mid F_u]$$

$$= Z_u . 1$$

Take X=max{S_T -K,0} ($\in$)F_T and let $A_t = E_Q[B_T^{-1} X \mid F_t]$ for t $\in$[0,T].

Note that for u<t, $A_u = E_Q[A_t \mid F_u]$:(=$E_Q[E_Q[B_T^{-1} X \mid F_t] \mid F_u]$)

So, A_t is also a martingale under Q.

From Martingale representation theorem,

there exists a predictable (pevisible) process ϕ={ϕ_t} such that

$dA_t = \phi_t dZ_t$.

Use this process ϕ={ϕ_t} to define a portfolio process V={V_t};

let $\psi_t = A_t - \phi_t Z_t$ and let $V_t \equiv V(\phi_t, \psi_t) = \phi_t S_t + \psi_t B_t$.

(Note that $A_t = B_t^{-1} V_t$, or $V_t = B_t A_t$: A_t is a discounted version of V_t)

Now, let us check that V_t is a self-financing portfolio by some calculation.

(self-financing :Def: $dV_t = \phi_t dS_t + \psi_t dB_t$.)

(note: in general, $dV_t = \phi_t dS_t + \psi_t dB_t + "S_t d\phi_t + d\phi_t dS_t + B_t d\psi_t + d\psi_t dB_t"$)

Start with $V_t = B_t A_t$.

$$dV_t = d(B_t A_t) = B_t dA_t + A_t dB_t \ ,$$

$$(\because dB_t dA_t = 0)$$

$$= B_t \phi_t dZ_t + (\phi_t Z_t + \psi_t) dB_t \ ,$$

$$(\because dA_t = \phi_t dZ_t \text{ and } A_t = \phi_t Z_t + \psi_t)$$

$$= \phi_t (B_t dZ_t + Z_t dB_t) + \psi_t dB_t$$

$$= \phi_t d(Z_t B_t) + \psi_t dB_t$$

$$= \phi_t dS_t + \psi_t dB_t \ .$$

Now, $V_T = B_T A_T = B_T(B_T^{-1}X) = X$

and $\{V_t, t \in [0,T]\}$ being self-financing

imply that under no-arbitrage assumption,

V_t copys the price of option

with pay-off X and that

V_0 must coincide with the price of the option at time zero.

$$V_0 = B_0 A_0 = A_0 = E_Q[B_T^{-1}X \mid F_0]$$

$$= E_Q[B_T^{-1}\max\{S_T\text{-}K,0\} \mid F_0]$$

$$= e^{-rT}E_Q[\max\{S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\tilde{W}_T}\text{-}K,0\}]$$

Design of options and derivatives

Pay-off function determines/identifies

the feature of option

Remarks for pay-off functions

: (1). Linearity of pay-off corresponds to linearity of prices.

: (2). For $h(S_T , B_T)$: has “linear homogeneity”;

$h(k \cdot S_T , k \cdot B_T) = k \cdot h(S_T , B_T)$,

It holds that

$h(S_t , B_t) = S_t \cdot (d/dS)h(St , B_t) + B_t \cdot (d/dB)h(S_t , B_t)$

PDE is linear. Also martingale defined is in a form of expectation that is linear

Hence option price must be linear combination, i.e. sum of the each solutions for

Pay-offs ; $S_T \cdot (d/dS)h(S_T , B_T)$ and $B_T \cdot (d/dB)h(S_T , B_T)$.

Exotics := not usual

Usual Options = Put and Call

Exotics (options and derivatives)

Path-independent

: Pay-later

: Chooser (path dependent?)

: Supershare

: max of two assets

Path-dependent

: Barrior

: Average (Asian)

: Parisian and Edokko

Journalism. Markets.

:1987. New York.

Portfolio Insurance (Protective Put).

:2005-2011. Tokyo.

Call(S:K)-3Put(S:K); S is USdollar/Jyen Currency rate.

K=100, S_0 =120(?). T=5 years. 120 down to 85.

(newspapers do not write details of contracts, except the above. Probably monthly cash-flows:pay-offs are included since the option holder are paying monthly.)

Section 2. Exotic derivatives

Based on “Nonparametric Statistics” of Brownian Motion.

: “Empirical distribution function” (Fixed level corridor).

: “Order statistics” (Brownian Quantiles, or α-quantiles).

: Ranks

: Focusing on designs and their ideas.

Section 2. Exotic derivatives Based on "Nonparametric Statistics" of Brownian Motion.

: (2-1).[Barrior options]

Barrior options and Edokko Barrior options.

(Brownian quantiles: Order Statistics of a path.)

: (2-2). [Corridors].

Stochastic Corridor (Rank Statistics.) and Fixed level Corridor (Empirical Distribution Functions)

: Forward contracts (to exchange Fixed and Stochastic).

: Stochastic Corridor: Spot one and forward starting one.

: Options on Stochastic Corridor.

: Emphasize an advantageous property of rank statistics.(distributional invariance).

: Try for an explicit form of a Forward price with some probability statement.

Standard (usual) Barrior Options.

(taken from a book Miura(2000) (in Japanese))

Standard (or, usual) Barrior Options

Let the price function , V(t, S_t).

Set a barrior a>0. a<S_0.

(Knock-out; V(τ,a)=0. $\tau = \inf\{t : S_t \leq$ a,0<t<T$\}$)

This V(t, S_t) saisfies: the following PDE in the area of "not knocked out ",

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S_t} rS_t + \frac{1}{2}\frac{\partial^2 C}{\partial {S_t}^2}\sigma^2 {S_t}^2 - rC = 0.$$

$(V(t,0) = 0.\ \lim_{S\to\infty} V(t,S) = S.)$

$$V(t,S_t) = E[e^{-r(\tau-t)}\max\{X_T - K, 0\} I\{\tau > T\} \mid X_t = S_t],$$ where we let a<K.

$$= \mathrm{C(t,S_{\it t})} - \left(\frac{S_t}{a}\right)^{\frac{2r}{\sigma^2}-1} C\left(t, \frac{a^2}{S_t}\right)$$

[Homework : Derive this pricing function by Calculating the expectation.]

Calculation.

Note: $E[e^{-r(\tau-t)}\max\{X_T-K,0\}\,|\,X_t=S_t]$

$=E[e^{-r(T-t)}\max\{X_T-K,0\}I\{\tau>T\}\,|\,X_t=S_t]+$

$E[e^{-r(\tau-t)}\max\{X_T-K,0\}I\{\tau<T\}\,|\,X_t=S_t]$

and the second term is calculated as;

$$\int_t^T E[e^{-r(u-t)}\max\{ae^{r(T-u)+\sigma(W_T-W_u)}-K,0\}\,|\,\tau=u]\;g(u:x,b)du,$$

where $\log S_t=x,$ and $\log a=b$

$$=\int_t^T C(T-u,a:K,r,\sigma)\;g(u:x,b)du.$$

Memo: probability density function of τ. 0<t< $\leq$ T. (note. P$\{\tau>T\}>0$)

$$g(t:x,b)=\frac{|x-b|}{\sigma\sqrt{2\pi t^3}}e^{-\frac{(x-b-\mu t)^2}{2\sigma^2 t}},\text{ where } \log S_t=x,\text{ and } \log a=b$$

We now record various probabilities associated with Brownian motion: $\mathbb{S}$

(6) $$P\left(\sup_{0\le s\le t} \mathbb{S}(s) > b\right) = 2P(N(0,t) > b)$$

$$= \int_0^t \frac{b}{\sqrt{2\pi s^3}} \exp\left(-\frac{b^2}{2s}\right) ds \qquad \text{for all } b > 0$$

$$= F_\tau(t)$$

where $\tau \equiv \inf\{s: \mathbb{S}(s) = b\}$; $\{\tau \le t\} \Leftrightarrow \{\sup_{0\le s\le t} \mathbb{S}(s) > b\}$

[Memo from last Wednesday.]

$$2P(N(0,t) > b) = 2\int_b^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \int_0^t \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} ds$$

by change of variable: s= $\frac{tb^2}{x^2}$, then, $-\frac{b}{2s^{\frac{3}{2}}} ds = \frac{1}{\sqrt{t}} dx$.

Edokko (Barrior) Options

From Fujita & Miura (2002)

: Idea is to wait for a while before knocking out,

rather than knocking out right away when S touches the barrior.

This will make a manipulation less easy (or difficult).

: Then, how long do we wait?

We need to define How it is determined?

EXAMPLE 2.2. *Delayed Barrier Option (Linetsky, 1999) = Cumulative Parisian Option (Chesney et al., 1997).*

This option is a down-and-out option that is knocked out when the occupation time below the barrier A exceeds a given fraction α, $0 < \alpha < 1$ of the maturity time T. Using our framework, for $\alpha (0 < \alpha < 1)$,

$$R_{K.O.} = \left\{ t \,\middle|\, \int_0^t 1_{(-\infty,A]}(S_u)du \geqq \alpha T \right\}.$$

In other words, we remark that the condition which the α percentile of the underlying asset $S_u (0 \leqq u \leqq T)$ becomes less than A is equivalent to this K.O. condition.

Equivalent to options with a K.O.condition ;

m(α:[0,T]) < A for a prefixed level A.; European type option.

Note that m(α:[0,t]), t $\in$[0,T] is a stochastic process.

Remark.

Regard that F(K)= $\frac{1}{t}\int_0^t I_{\{-\infty,K\}}(S_u)du$ is a stochastic process F(t: K) with time t.

We already know P{F(t: K) $< x$} for x $\in$[0,1].

Since F(t: K) is nonnegative and

has continuous and nondecreasing path in t,

we may easily define a first hitting time of F(t: K) to a level x.

Another Remark.

Brownian quantile is not Markov ?.

[Homework. Discuss if a Brownian quantile is Markov.]

Look at m($\alpha:[0,t]$). Given it at time t, can we know m($\alpha:[0,t+dt]$) only with the information given at time t and with S_u , u $\in$[t,t+dt] ?

Note. As for Asian option, the average A(S:[0,t]) can be treated as Markov by making two dimentional stochastic process (S_t, A(S:[0,t]).

Therefore, Asian option can be discissed with PDE approach (for two dementional stochastic process).

EXAMPLE 2.3. *Cumulative Parisian Edokko Option.*

This option is a down-and -out option that is knocked out when the occupation time below the barrier A exceeds a given fraction $\alpha, 0 < \alpha < 1$ of the remaining caution time $T - \tau_A$. Using our framework, for $\alpha(0 < \alpha < 1)$,

$$R_{K.O.} = \left\{ t \middle| \int_{\tau_A}^{t} 1_{(-\infty,A]}(S_u)du \geqq \alpha(T - \tau_A) \right\}.$$

REMARK 2.1. $\frac{\int_{\tau_A}^{t} 1_{(-\infty,A]}(S_u)du}{T-\tau_A} \geqq \alpha \iff$ *α-percentile of $S_u(\tau_A \leqq u \leqq T) \leqq A$.*

In other words, we remark that the condition which α percentile of the underlying asset $S_u(\tau_A \leqq u \leqq T)$ becomes less than A is equivalent to this K.O. condition.

EXAMPLE 2.4. *Parisian Option (Chesney et al., 1997).*

A Parisian option becomes worthless if the underlying asset reaches a prespecified level A and remains continuously below this level for a time interval longer than a fixed number D. Specifying $R_{K.O.}$, for a positive constant D, $R_{K.O.} = \{t|$the length of the current excursion below under the level A straddling $t \geqq D\}$.

EXAMPLE 2.5. *Parisian Edokko Option.*

A Parisian Edokko option becomes worthless if the underlying asset reaches a prespecified level A and remains continuously below this level for a time interval longer than a fixed number $\alpha(T - \tau_A)$ for $\alpha(0 < \alpha < 1)$. Specifying $R_{K.O.}$, for $\alpha(0 < \alpha < 1)$, $R_{K.O.} = \{t|\text{the length of the current excursion below under the level } A \text{ straddling } t \geqq \alpha(T - \tau_A)\}$.

Parisian <=> Remains CONTINUOUSLY under the level A

for more than a prefixed amount of time D.

Edokko <=> Remain under the level A

for α-portion of th remaining time $[\tau_A , T]$

Pricing of Cumulative Parisian Edokko option.

Use the probability distribution we have already discussed.

3. Pricing

We can obtain closed form expressions of the prices of the above-mentioned examples in Black Scholes model. In this section, choosing 'cumulative Parisian Edokko [illegible]

[illegible]

Let $X(t)$ be a continuous stochastic process.
We put $A_X(t, x) = \frac{1}{t}\int_0^t 1_{(-\infty,x]}(X(s))ds$, where

$$1_{(-\infty,x]}(y) = \begin{cases} 1 & \text{if } y \leqslant x \\ 0 & \text{if } y \geqslant x \end{cases}.$$

Since $A_X(t, \cdot)$ is increasing, the inverse function $m_X(t, \cdot)$ exists i.e.

$$A_X(t, m_X(t, \alpha)) = \alpha \; (0 < \alpha < 1), \; m_X(t, A_X(t, x)) = x,$$

$$A_X(t, x) > \alpha \rightleftharpoons m_X(t, \alpha) < x$$

hold. Miura (1992) called options related to $m_X(t, \alpha)$ *α-percentile options.*
[illegible]

hold. Miura (1992) called options related to $m_X(t,\alpha)$ *α-percentile options*. Seeing that $m_X(t,1/2)=$ the median of $X(s)\,(0 \leqslant s \leqslant t)$ and $m_X(t,1-0) = \max_{0\leqslant s\leqslant t} X(s)$, we can observe that *$\alpha$-percentile options* are based on order statistics and have merits that are hardly affected by extreme values. For pricing of *α-percentile options*, see Akahori (1995), Dassios (1995), Embrechts et al. (1995), Fujita (1997, 2000), and Yor (1995). We use this α-percentile as stopping conditions of derivative contracts. In this sense, we may call Example 2.2 and Example 2.3 *α-percentile barrier options*.

Let W_t denote a standard Brownian motion. First we prepare the following theorem about the joint density of Brownian motion and its occupation time. This formula is obtained by Fujita (1997) to price the α-percentile option with a payoff $\max(S_T - m_S(T,\alpha), 0)$. This result is equivalent to an occupation time law

THEOREM 3.1.

$$P(W_t \in da, \int_0^t 1_{(-\infty,0]}(W_s)ds \in du)$$

$$= \begin{cases} (\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2(t-s)}} ds) da du \cdots \text{for } a > 0 \\ (\int_0^u \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2s}} ds) da du \cdots \text{for } a < 0. \end{cases}$$

Proof. We put that $f(t,x) = E[1_{[a,+\infty)}(x+W_t)e^{-\beta\int_0^t 1_{(-\infty,0]}(x+W_s)ds}]$ (*for* $a > 0, \beta > 0$).

Using the Feynman-Kac Theorem, we have:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} - \beta 1_{(-\infty,0]}(x) f \qquad f(0,x) = 1_{[a,+\infty)}(x).$$

Taking Laplace transforms of both sides, and denoting: $\hat{f}(\xi, x) = \int_0^{+\infty} dt e^{-\xi t} f(t,x)$, we obtain:

$$-1_{[a,+\infty)}(x) + \xi \hat{f} = \frac{1}{2}\frac{\partial^2 \hat{f}}{\partial x^2} - \beta 1_{(-\infty,0]}(x)\hat{f}.$$

Solving this ordinary differential equation and considering boundary conditions at 0 and a, we obtain

$$\hat{f}(0) = \frac{e^{-\sqrt{2\xi}a}}{\sqrt{\xi}(\sqrt{\xi} + \sqrt{\xi+\beta})}.$$

Then, we see

$$\frac{-\partial \hat{f}(0)}{\partial a} = \sqrt{2}\frac{e^{-\sqrt{2\xi}a}}{\sqrt{\xi} + \sqrt{\xi+\beta}}$$

Solving this ordinary differential equation and considering boundary conditions at 0 and a, we obtain

$$\hat{f}(0) = \frac{e^{-\sqrt{2\xi}a}}{\sqrt{\xi}(\sqrt{\xi} + \sqrt{\xi + \beta})}.$$

Then, we see

$$\begin{aligned}
\frac{-\partial \hat{f}(0)}{\partial a} &= \sqrt{2}\frac{e^{-\sqrt{2\xi}a}}{\sqrt{\xi} + \sqrt{\xi + \beta}} \\
&= \left(\widehat{\frac{e^{-\beta t} - 1}{(-\beta)\sqrt{2\pi t^3}}}\right)\left(\widehat{\frac{\sqrt{2}a}{2\sqrt{\pi t^3}}e^{-\frac{(\sqrt{2}a)^2}{4t}}}\right) \\
&= \left(\widehat{\frac{1 - e^{-\beta t}}{\sqrt{2\pi t^3}\beta} * \frac{a}{\sqrt{2\pi t^3}}e^{-\frac{a^2}{2t}}}\right) \\
&= \left(\widehat{\int_0^t \frac{a}{\sqrt{2\pi s^3}}\frac{1 - e^{-\beta s}}{\beta}\frac{a}{\sqrt{2\pi (t - s)^3}}e^{-\frac{a^2}{2(t-s)}}ds}\right) \\
&= \left(\int_0^t e^{-\beta u}du \int_u^t \widehat{\frac{a}{2\pi\sqrt{s^3(t - s)^3}}e^{-\frac{a^2}{2(t-s)}}ds}\right).
\end{aligned}$$

This shows that for $a > 0$, $P(W_t \in da, \int_0^t 1_{(-\infty,0]}(W_s)ds \in du) = (\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2(t-s)}} ds)dadu$. Similarily we obtain the joint density function for $a < 0$.

REMARK 3.1.

Chesney et al. (1997) got the same results by another approach. Also Karatzas and Shreve (1991, p. 423, Prop. 3.9) obtained the similar results of this Theorem.

We denote the joint density function of $(W_t, \int_0^t 1_{(-\infty,0]}(W_s)ds)$ by $f_{(W_t, \int_0^t 1_{(-\infty,0]}(W_s)ds)}(a, u)$.

Applying Girsanov's theorem, we get that the joint density function $g_{(X_t^{\mu,\sigma}, \int_0^t 1_{(-\infty,0]}(X_s^{\mu,\sigma})ds)}(a, x)$ for a Brownian motion with drift $(\sigma W_t + \mu t = X_t^{\mu,\sigma})$ is:

$$g_{(X_t^{\mu,\sigma}, \int_0^t 1_{(-\infty,0]}(X_s^{\mu,\sigma})ds)}(a, x) = e^{-\frac{\mu^2 t}{2\sigma^2}} e^{\frac{\mu a}{\sigma^2}} (1/\sigma) f_{(W_t, \int_0^t 1_{(-\infty,0]}(W_s)ds)} \left(\frac{a}{\sigma}, x\right).$$

From we can determine the price of Cumulative Parisian Edokko Option under the Black-Scholes model.

From we can determine the price of Cumulative Parisian Edokko Option under the Black-Scholes model.

Under the risk neutral measure in the Black-Scholes model, we take the S.D.E. which the underlying asset price $S(t)$ satisfies as follows:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = S,$$

where r = the instantaneous risk free rate, σ = the volatility.

Then we know that

$$\begin{aligned} S_t &= Se^{\sigma W_t + (r - \frac{1}{2}\sigma^2)t} \\ &= Se^{X^{r-\frac{1}{2}\sigma^2, \sigma}(t)}. \end{aligned}$$

We denote a payoff at maturity T as $f(S_T)$. Considering stopping condition $1_{m_X(T,\alpha)\leqq A}$, we have that a payoff of Cumulatine Parisian Edokko Option at the maturity T $(1 - 1_{m_X(T-\tau_A,\alpha)\leqq A})f(S_T)$ where we assume that $A < S$. Then the price of Cumulative Parisian Edokko Option ($= C(T, S, \alpha, x)$) is obtained by $C(T, S, \alpha, A) = E(e^{-rT}(1 - 1_{m_X(T-\tau_A,\alpha)\leqq A})f(S_T))$.

Remarking that $E(e^{-rT} f(S_T)) = e^{-rT} \int_{-\infty}^{+\infty} f(Se^{(r-(1/2)\sigma^2)T+\sigma x}) \frac{e^{(-1/2)x^2}}{\sqrt{2\pi}} dx = C_1(T, S) =$ usual $B.S.$, it is enough to calculate that $E(e^{-rT} 1_{m_X(T-\tau_A,\alpha)\leqq x} f(S_T)) = C_2(T, S, \alpha, x)$.

So, it is sufficient to obtain the joint density function of $(S_T, \int_0^T 1_{(-\infty,A]}(S_s)ds, \tau_A)$.

$$\begin{aligned}
&\left(S_T, \int_0^T 1_{(-\infty,A]}(S_s)ds\right) \\
&= \left(Se^{(r-(1/2)\sigma^2)T+\sigma W_T}, \int_0^T 1_{(-\infty,A]}(Se^{(r-(1/2)\sigma^2)s+\sigma W_s})ds\right) \\
&= \left(Se^{X_T^{(r-(1/2)\sigma^2),\sigma}}, \int_0^T 1_{(-\infty,\log(A/S)]}(X_s^{(r-(1/2)\sigma^2),\sigma})ds\right).
\end{aligned}$$

We put $\tau_A = \inf\{t | S_t = A\} = \inf\{t | X_t^{(r-(1/2)\sigma^2),\sigma} = \log A/S\}$. Then, conditioning $\tau_A = u$, we have

$$
\begin{aligned}
&\left(S_T, \int_0^T 1_{(-\infty,A]}(S_s)ds \right)\Big|_{\tau_A=u} \\
&= \left(Ae^{X_T^{(r-(1/2)\sigma^2),\sigma} - X_u^{(r-(1/2)\sigma^2),\sigma}}, \int_u^T 1_{(-\infty,A]}(Ae^{X_s^{(r-(1/2)\sigma^2),\sigma} - X_u^{(r-(1/2)\sigma^2),\sigma}})ds \right)\Big|_{\tau_A=u} \\
&= \left(Ae^{\hat{X}_{T-u}^{(r-(1/2)\sigma^2),\sigma}}, \int_u^T 1_{(-\infty,0]}(\hat{X}_{s-u}^{(r-(1/2)\sigma^2),\sigma})ds \right)\Big|_{\tau_A=u} \\
&= \left(Ae^{\hat{X}_{T-u}^{(r-(1/2)\sigma^2),\sigma}}, \int_0^{T-u} 1_{(-\infty,0]}(\hat{X}_s^{(r-(1/2)\sigma^2),\sigma})ds \right)\Big|_{\tau_A=u},
\end{aligned}
$$

where we put $\hat{X}_t = X_t - X_u$ and we remark that $\hat{X}_t$ is independent $\mathcal{F}_u = \sigma\{X_s; s \leqq u\}$.

So, we see that

$$
\begin{aligned}
C_2(T, S, \alpha, A) &= E(e^{-rT} 1_{m_S(T-\tau_A,\alpha)\leqq A} f(S_T)) \\
&= e^{-rT} E(E(1_{m_S(T-\tau_A,\alpha)\leqq A} f(S_T)|\tau_A)) \\
&= e^{-rT} \int_0^T E(1_{m_S(T-u,\alpha)\leqq A} f(S_T)|\tau_A = u) h_{\tau_A}(u) du \\
&= e^{-rT} \int_0^T \int \int_{b \geqq A(T-u)} f(Ae^a, b) g_{(X_{T-u}^{(r-(1/2)\sigma^2),\sigma}, \int_0^{T-u} 1_{(-\infty,0]}(X_s^{(r-(1/2)\sigma^2),\sigma}) ds)} \\
&\quad (a, b) da db h_\tau(u) du,
\end{aligned}
$$

where we recall the known result $h_{\tau_A}(s)$ = the density function of $h_{\tau_A} = \frac{|\log A/S|}{\sigma\sqrt{2\pi s^3}} e^{\frac{-(\log A/S-(r-(1/2)\sigma^2)s)^2}{2s\sigma^2}}$.

That is, the Price of Cumulative Parisian Edokko Option = B.S. − $C_2(T, S, \alpha, A)$.

Stochastic Corridor

:Fixed level corridor.

: Stochastic corridor.

: Forward-starting Corridors (a fixed one and a stochastic one)

: Forward contract to Exchange the above two.

(for a situation where a suitable K is not easily determined.)

: Options on fixed or stochastic corridors.

Prices of these derivatives can be calculated in a straight forward manner in our case. However, in case where S follows Levy process or any other useful process, other than Geometric Brownian motion, I do not know if we can calculate it. Ornstein-Uhrenbeck process => first hitting time seems discussed.

Fixed level corridor (or, fixed corridor for short.)

$$S_u = S_0 e^{X_u} = S_0 e^{\mu u + \sigma W_u}, \text{ for } u \in [0, T], \tag{1}$$

where S_0 is the random initial price and W_u is a standard Brownian motion with zero mean.

Now, the *corridor* is defined as follows. For any fixed constant K, let

$$F(K) = \frac{1}{T} \int_0^T I\{S_u \leq K\} du. \tag{2}$$

This is just a continuous-time version of the empirical process for stock prices during the time interval $[0, T]$. It is a measure of the proportion of time the stock prices stay below the given fixed value K during the time interval $[0, T]$. We will call it a fixed-level corridor or fixed corridor for short.

This quantity depends on the path of stock price, and it can be determined only at the end of the time interval. For example, consider an application in discrete-time setting such as the currency exchange rate derivatives. This statistic counts the proportion of days the exchange rate stays below the given fixed level K, and the *pay-off* (which is the value of the derivative at the time of exercise, or of expiration) of the derivative (contract) may promise to pay to the holder of the derivative the amount of money proportional to the statistic. This is called a *corridor option*. These corridors could also be used in principle for other applications such as weather derivatives to count the number of days where the daily-temperature stays below a fixed level.

In this chapter, we define and study the properties of a new derivative called *stochastic corridor*. Specifically, consider a fixed day t with stock price S_t, which is random. Define the rank process

$$R(t) = \frac{1}{T}\int_0^T I\{S_u \le S_t\}du. \tag{4}$$

This has a similar interpretation as the fixed corridor except that the fixed value of K in the fixed corridor has been replaced by S_t which is stochastic. Note that $R(t)$ does not depend on S_0 since

$$I\{S_u \le S_t\} = I\{S_0e^{X_u} \le S_0e^{X_t}\} = I\{X_u \le X_t\}.$$

The fixed-level corridor and the stochastic-level corridor both can be used as payoff of derivatives.

The following lemma plays a key role at several parts in this chapter where a calculation is encountered for an expectation of the nonparametric statistics such as a fixed corridor or a stochastic corridor. The proof can be found in Fujita(1997) or Fujita & Miura(2002,2004). See also the handbook Borodin & Salminen(2002) for the result without proof.

Lemma 1.

$$P(W_t \in da, \int_0^t I\{W_s < 0\}ds \in du) = \left(\int_u^t \left(\frac{a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2(t-s)}} ds\right) da \quad du,\right.$$

for $a > 0$.

$$P(W_t \in da, \int_0^t I\{W_s < 0\}ds \in du) = \left(\int_0^u \left(\frac{-a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2s}} ds\right) da \quad du \text{ for}\right.$$

$a < 0$.

4 Corridor Swap

The fixed corridor $F(K)$ and the stochastic corridor $R(t)$ both can be used separately as payoff of derivatives. Their prices at time 0 are given respectively by

$$e^{-rT}E_0[F_{K,T}^{T,\sigma}], \qquad e^{-rT}E_0[R_{t,T}^{T,\sigma}]$$

in a Black-Scholes market.

We go further to define a "swap" or an exchange of the two derivatives which requires appropriate choice of the value of K. The payoff of the swap contract is $F(K) - R(t)$. This price at time of the contract is zero so that we have, as usual,

$$0 = e^{-rT}E_0[\int_0^T I\{S_u \leq K\}du - \int_0^T I\{S_u \leq S_t\}du]$$

Thus, the constant K has to be chosen to satisfy the equation

$$E_0[\int_0^T I\{S_u \leq K\}du] = E_0[\int_0^T I\{S_u \leq S_t\}du].$$

Note that righthand-side is a non-negative bounded constant less than T, and the lefthand-side is a strictly increasing continuous function of K ranging from zero to T. So there must exist a constant K which satisfies the above equality.

It is necessary to have an explicit functional form of these expectations in order to obtain the numerical value of K. They can be obtained by using the distributional results in Section 2.

5 Corridor Option

It is possible to define Put-type and Call-type options using the fixed and stochastic corridors. Their pricing can be done in a straightforward manner since it does not require any further distributional results.

We define a corridor call option on the stochastic corridor with the fixed level corridor as its exercise value. The pay-off of the corridor call option is $V_{C,T} = \max(R(t) - F(K),\ 0)$. Similarly, the pay-off of the corridor put option is $V_{P,T} = \max(F(K) - R(t),\ 0)$. The prices of these Call and Put at time zero in the Black-Scholes model are given by $V_{C,0} = e^{-rT}E_0[V_{C,T}]$ and $V_{P,0} = e^{-rT}E_0[V_{P,T}]$ respectively. The expectation for Call option can be calculated as follows.

Theorem 1.

$$V_{C,0} = e^{-rT} E_0[B^{r,\sigma}] = e^{-rT} E_0[e^{\frac{r}{\sigma} W_T - (\frac{r}{\sigma})^2 \frac{T}{2}} B^{0,\sigma}]$$

where

$$\begin{aligned} B^{r,\sigma} &= B_1^{r,\sigma} + B_2^{r,\sigma} \\ &= B_{1,1}^{r,\sigma} + B_{1,2}^{r,\sigma} + B_{2,1}^{r,\sigma} + B_{2,2}^{r,\sigma} \end{aligned}$$

and

$$B^{0,\sigma} = B_{1,1}^{0,\sigma} + B_{1,2}^{0,\sigma} + B_{2,1}^{0,\sigma} + B_{2,2}^{0,\sigma},$$

where

$$\begin{aligned} B^{\mu,\sigma} &= \max\{\int_0^T I\{S_u \leq S_t\} du - \int_0^T I\{S_u \leq K\} du, 0\} \\ &= \int_0^T I\{S_u \leq S_t\} du \cdot I\{S_t > K\} - \int_0^T I\{S_u \leq K\} du \cdot I\{S_t > K\} \\ &\triangleq B_1^{\mu,\sigma} + B_2^{\mu,\sigma} \end{aligned}$$

$$
\begin{aligned}
&= \int_t^T I\{S_u \le S_t\}du \cdot I\{S_t > K\} - \int_t^T I\{S_u \le K\}du \cdot I\{S_t > K\} \\
&\quad + \int_0^t I\{S_u \le S_t\}du \cdot I\{S_t > K\} - \int_0^t I\{S_u \le K\}du \cdot I\{S_t > K\} \\
&\triangleq B_{1,1}^{\mu,\sigma} + B_{1,2}^{\mu,\sigma} + B_{2,1}^{\mu,\sigma} + B_{2,2}^{\mu,\sigma}
\end{aligned}
$$

The terms above are calculated in the following lemmas.

Lemma 2.

$$\begin{aligned}
E_0[B_1^{r,\sigma}] &= E_0[\{\int_t^T I\{S_u \le S_t\}du - \int_t^T I\{S_u \le K\}du\} \cdot I\{S_t > K\}] \\
&= E_0[e^{\frac{r}{\sigma}Z_{T-t}-(\frac{r}{\sigma})^2\frac{T-t}{2}} \int_0^{T-t} I\{Z_v \le 0\}dv] \cdot E_0[I\{W_t > A^{r,\sigma}\}] \\
&\quad - \int_{\{S_0 e^{rt+\sigma w} > K\}} \{E_0[e^{\frac{r}{\sigma}Z_{T-t}-(\frac{r}{\sigma})^2\frac{T-t}{2}} \int_0^{T-t} I\{Z_v \le rt \\
&\quad +\sigma w + \log\frac{K}{S_0}\}dv \,|W_t = w]\}n(w:0,t)dw
\end{aligned}$$

Proof. Define $A^{r,\sigma} = \frac{1}{\sigma}(\log\frac{K}{S_0} - rt))$, and $Z_{u-t} = W_u - W_t$. Note that W_t and $W_u - W_t$ are independent.

$$\begin{aligned}
B_{1,1}^{r,\sigma} &= \int_t^T I\{S_u \le S_t\}du \cdot I\{S_t > K\} \\
&= \int_t^T I\{r(u-t) + \sigma(W_u - W_t) \le 0\}du \cdot I\{W_t > A^{r,\sigma}\} \\
&= \int_0^{T-t} I\{ru + \sigma Z_u \le 0\}du \cdot I\{W_t > A^{r,\sigma}\} \\
B_{1,2}^{r,\sigma} &= \int_t^T I\{S_u \le K\}du\} \cdot I\{S_t > K\} \\
&= \int_t^T I\{S_0 e^{ru+\sigma W_u} \le K\}du\} \cdot I\{S_0 e^{rt+\sigma W_t} > K\} \\
&= \int_t^T I\{r(u-t) + \sigma(W_u - W_t) \le -rt - \sigma W_t + \log\frac{K}{S_0}\}du \\
&\quad \times I\{W_t > A^{r,\sigma}\} \\
&= \int_0^{T-t} I\{rv + \sigma Z_v \le rt + \sigma W_t + \log\frac{K}{S_0}\}dv \cdot I\{W_t > A^{r,\sigma}\}. \\
&\quad \text{(where } v = u - t\text{)}
\end{aligned}$$

In the last step, note that $\log\left(\frac{K}{S_t}\right) = -rt - \sigma W_t + \log\frac{K}{S_0} < 0$ in $\{W_t > A^{r,\sigma}\} = \{S_t > K\}$. Now,

$$\begin{aligned}
E_0[B_1^{r,\sigma}] &= E_0[B_{1,1}^{r,\sigma}] + E_0[B_{1,2}^{r,\sigma}] \\
&= E_0[e^{\frac{r}{\sigma}Z_{T-t}-(\frac{r}{\sigma})^2\frac{T-t}{2}}\int_0^{T-t} I\{Z_v \le 0\}dv]\cdot E_0[I\{W_t > A^{r,\sigma}\}] \\
&\quad - \int_{\{S_0e^{rt+\sigma w}>K\}}\{E_0[e^{\frac{r}{\sigma}Z_{T-t}-(\frac{r}{\sigma})^2\frac{T-t}{2}}\int_0^{T-t} I\{Z_v \le \frac{1}{\sigma}(rt \\
&\quad +\sigma w + \log\frac{K}{S_0})\}dv\,|W_t = w]\,\}n(w:0,t)dw \\
&= E_0[e^{\frac{r}{\sigma}Z_{T-t}-(\frac{r}{\sigma})^2\frac{T-t}{2}}\int_0^{T-t} I\{Z_v \le 0\}dv]\cdot E_0[I\{W_t > A^{r,\sigma}\}] \\
&\quad - \int_{\{S_0e^{rt+\sigma w}>K\}}\{E_0[e^{\frac{r}{\sigma}Z_{T-t}-(\frac{r}{\sigma})^2\frac{T-t}{2}}(\tau \\
&\quad + \int_\tau^{T-t} I\{Z_v \le 0\}dv)\,|W_t = w]\,\}n(w:0,t)dw,
\end{aligned}$$

where $\tau = \inf\{v : Z_v \ge A^*, 0 \le v \le T\}$, $A^* = \frac{1}{\sigma}(rt + \sigma w + \log\frac{K}{S_0})$ and $n(w:0,t)$ is the density of the normal distribution with mean zero and variance t.

□

Lemma 3.

$$
\begin{aligned}
E_0[B_2^{r,\sigma}] &= E_0[\{\int_0^t I\{S_u \le S_t\}du - \int_0^t I\{S_u \le K\}du\} \cdot I\{S_t > K\}] \\
&= E_0[e^{\frac{r}{\sigma}Z_t - (\frac{r}{\sigma})^2\frac{t}{2}} \int_0^t I\{0 \le Z_v\}dv \cdot I\{S_0 e^{Z_t} > K\}] \\
&\quad - \int_0^t E_0[e^{\frac{r}{\sigma}(Z_{t-s} + \frac{1}{\sigma}\log(\frac{K}{S_0})) - (\frac{r}{\sigma})^2\frac{t}{2}}(s + \int_0^{t-s} I\{Z_v \le 0\}dv) \\
&\quad \times I\{Z_{t-s} > 0\} \,|\tau = s]g(s)ds
\end{aligned}
$$

Proof. We rely on Cameron-Martin theorem to reduce the calculations for the case $\mu \neq 0$ to that for the case $\mu = 0$.

$$
\begin{aligned}
E_0[B^{r,\sigma}_{2,1}] &= E_0[\{\int_0^t I\{S_u \le S_t\}du \cdot I\{S_t > K\}] \\
&= E_0[\int_0^t I\{S_0 e^{ru+\sigma W_u} \le S_0 e^{rt+\sigma W_t}\}du \cdot I\{S_0 e^{rt+\sigma W_t} > K\}] \\
&= E_0[\int_0^t I\{0 \le rv + \sigma Z_v\}dv \cdot I\{S_0 e^{rt+\sigma Z_t} > K\}] \\
&\quad (\text{where } Z_v = W_t - W_{t-v}, \text{ and note that } Z_t = W_t) \\
&= E_0[e^{\frac{r}{\sigma}Z_t - (\frac{r}{\sigma})^2 \frac{t}{2}} \int_0^t I\{0 \le Z_v\}dv \cdot I\{S_0 e^{Z_t} > K\}] \\
&= \iint_{\log(\frac{K}{S_0})<x<\infty, 0<y<t} e^{\frac{r}{\sigma}x - (\frac{r}{\sigma})^2 \frac{t}{2}} y f_{(Z_t, \int_0^t I\{Z_s \le 0\}ds)}(x,y)dydx.
\end{aligned}
$$

$$
\begin{aligned}
E_0[B_{2,2}^{r,\sigma}] &= E_0[\{\int_0^t I\{S_u \le K\}du\} \cdot I\{S_t > K\}] \\
&= E_0[\{\int_0^t I\{\frac{r}{\sigma}u + W_u \le \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\}du\} \cdot I\{\frac{r}{\sigma}t + W_t > \frac{1}{\sigma}\log(\frac{K}{S_0})\}] \\
&= E_0[e^{\frac{r}{\sigma}W_t - (\frac{r}{\sigma})^2\frac{t}{2}} \int_0^t I\{W_u \le \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right)\}du \cdot I\{W_t > \frac{1}{\sigma}\log(\frac{K}{S_0})\}] \\
&= E_0[e^{\frac{r}{\sigma}W_t - (\frac{r}{\sigma})^2\frac{t}{2}}(\tau + \int_\tau^t I\{W_u - W_\tau \le 0\}du) \cdot I\{W_t > \frac{1}{\sigma}\log(\frac{K}{S_0})\}] \\
&\quad (\text{where } \tau = \inf\{u : W_u > \frac{1}{\sigma}\log(K/S_0), 0 < u < t\}.)
\end{aligned}
$$

$$= E_0[e^{\frac{r}{\sigma}(Z_{t-\tau}+\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right))-\left(\frac{r}{\sigma}\right)^2\frac{t}{2}}(\tau + \int_0^{t-\tau} I\{Z_v \leq 0\}dv) \cdot I\{Z_{t-\tau} > 0\}]$$

$$= \int_0^t E_0[e^{\frac{r}{\sigma}(Z_{t-s}+\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right))-\left(\frac{r}{\sigma}\right)^2\frac{t}{2}}(s + \int_0^{t-s} I\{Z_v \leq 0\}dv)$$

$$\times I\{Z_{t-s} > 0\} \,|\tau = s]g(s)ds$$

(where $Z_v = W_{\tau+v} - W_\tau, g(\cdot)$ is the probability density function of τ.

Note $Z_{t-\tau} = W_t - W_\tau = W_t - \frac{1}{\sigma}\log\left(\frac{K}{S_0}\right))$

$$= \int_{0<s<t} \iint_{0<x<\infty, 0<t-s} e^{\frac{r}{\sigma}(x+\frac{1}{\sigma}\log\left(\frac{K}{S_0}\right))-\left(\frac{r}{\sigma}\right)^2\frac{t}{2}}(s+y)$$

$$\times f_{(Z_{t-s}, \int_0^{t-s} I\{Z_v \leq 0\}dv)(x,y)} g(s)dxdyds$$

$\square$

Put-Call Parity

For any random variables X and Y, we have an equality; $\max\{X - Y, 0\} + Y = X + \max\{Y - X, 0\}$. Applying this relation to our Call and Put regarding X and Y as our stochastic corridor $R(t)$ and fixed level corridor $F(K)$,

$$\max\left[\int_0^T I\{S_u \leq S_t\}du - \int_0^T I\{S_u \leq K\}du, \;\; 0\right] + \int_0^T I\{S_u \leq K\}du$$
$$= \int_0^T I\{S_u \leq S_t\}du + \max\left[\int_0^T I\{S_u \leq K\}du - \int_0^T I\{S_u \leq S_t\}du, \;\; 0\right]$$

Since the pay-off of the left hand side and the right hand side of the equation coincide, the prices at time zero of the derivatives corresponding to each side must be equal under the assumption that the market does not allow any arbitrage. Hence, using the linearity of expectation, we have (the price of corridor call)+(price of fixed corridor) =(the price of stochastic corridor)+(price of corridor put)

6 Forward Starting Corridor

Let $[T_0, T_1]$, $0 < T_0 < T_1$, be a future time interval where a corridor option counts the amount of time that the stock prices stay below a level, either fixed or stochastic. Now, the payoffs of forward starting fixed corridor and stochastic corridor are respectively,

$$F(K, (T_0, T_1)) = \int_{T_0}^{T_1} I\{S_u \leq K\} du$$

and

$$R(t, (T_0, T_1)) = \int_{T_0}^{T_1} I\{S_u \leq S_t\} du.$$

A contract is made at time 0 and the payoff is paid to the holder at time T_1. Then the prices of these options in the Black-Scholes model are

$$e^{-rT_1} E_0[\int_{T_0}^{T_1} I\{S_u \leq K\} du]$$

$$e^{-rT_1} E_0[\int_{T_0}^{T_1} I\{S_u \leq S_t\} du]$$

respectively.

As we saw in the previous section that the probability distribution of the stochastic corridor is independent of the value S_{T_0}, the value of the initial stock price in the future time interval $[T_0, T_1]$. This independence property may be expected to be useful in practice when they set a level for the fixed corridor. In order to decide a constant level K, it may be required in practice to have a certain idea or a prediction of overall level of stock prices during the future time interval $[T_0, T_1]$. Since it is not easy to make a prediction, it may be plausible sometimes to depend on a stochastic value to determine an overall level, for example S_{T_0}. Or there might be a special time point t during the future time interval $[T_0, T_1]$ that is suitable for making S_t the stochastic level for the stochastic corridor.

If one wants to compensate the result from the ambiguity of a suitable value of K with the difference between the two forward starting corridors, one can swap to exchange the forward starting fixed corridor with the forward starting stochastic corridor. The payoff of this swap is

$$\int_{T_0}^{T_1} I\{S_u \leq S_t\}du - \int_{T_0}^{T_1} I\{S_u \leq K\}du$$

or

$$\int_{T_0}^{T_1} I\{S_u \leq K\}du - \int_{T_0}^{T_1} I\{S_u \leq S_t\}du.$$

As in Section 3, we need to be able to determine a proper theoretical value of K which makes the price of the swap contract be zero at the time of the contract, i.e. at time 0. That is, K has to satisfy the equation

$$0 = e^{-rT_1} E_0\left[\int_{T_0}^{T_1} I\{S_u \leq K\}du - \int_{T_0}^{T_1} I\{S_u \leq S_t\}du\right]$$

In other words,

$$E_0\left[\int_{T_0}^{T_1} I\{S_u \leq K\}du\right] = E_0\left[\int_{T_0}^{T_1} I\{S_u \leq S_t\}du\right].$$

The above expectations are the conditional expectations taken under the condition that the value of S_0 is given. The existence of such a constant K is assured using the same argument as in the previous section.

The probability distribution of $\int_{T_0}^{T_1} I\{S_u \leq S_t\}du$ is S_{T_0}- independent and is the same as that of $\int_0^{T_1-T_0} I\{S_u \leq S_t\}du$. (See Fujita and Miura(2004)). However, the calculation for $E_0[\int_{T_0}^{T_1} I\{S_u \leq K\}du]$ requires some additional comments.

$$
\begin{aligned}
E_0[\int_{T_0}^{T_1} I\{S_u \leq K\}du\,|S_0] &= E_0[E_{T_0}[\int_{T_0}^{T_1} I\{S_u \leq K\}du\,|S_{T_0}]\,|S_0] \\
&= E_0[E_{T_0}[\int_{T_0}^{T_1} I\{e^{X_u - X_{T_0}} \leq \frac{K}{S_{T_0}}\}du\,|S_{T_0}]\,|S_0].
\end{aligned}
$$

Since for any u in $[T_0, T_1]$, $(X_u - X_{T_0})$ and X_{T_0} or equivalently S_{T_0} are stochastically independent of each other, the expectation inside can be calculated with any given value of S_{T_0} and the result integrated with respect to the density function of S_{T_0}. So this does not involve a joint distribution function.

A call option with a payoff

$$\max\left[\int_{T_0}^{T_1} I\{S_u \leq S_t\}du - \int_{T_0}^{T_1} I\{S_u \leq K\}du, \quad 0\right],$$

is possible. Its price can be calculated in a similar way to that for the option for the spot starting corridor.

Section 3. Weather Derivatives

: Underlying variables are not tradable. Incomplete Market models.

: Taking from Mark Davis's papers.

: Marginal rate of substitutions.

: How about a forward starting Weather derivatives; conditionally more accurate forcasting of the weather (?).

Can we use rank to use an advantage of rank statistics.

[2] Davis M H A 1998 Option pricing in incomplete markets *Mathematics of Derivative Securities* ed M A H Dempster and S R Pliska (Cambridge: Cambridge University Press)

Option Pricing in Incomplete Markets

Mark H.A. Davis

Abstract

In this chapter a general option pricing formula is proposed, using arguments based on marginal substitution value. By giving the investor an external objective in the form of a utility maximization problem we arrive at a unique price in situations where standard arbitrage arguments cannot be used. Further, we show using Markov process theory that the price can be expressed as a discounted expectation where both the measure and the discount rate are uniquely determined. Models with stochastic coefficients and transaction cost models are studied in detail.

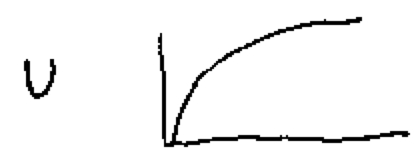
1 A General Option Pricing Formula

The Black-Scholes option pricing formula depends on exact replication and is only applicable in complete markets. It expresses the option value as the expected discounted exercise value where the expectation is calculated using the uniquely defined "martingale measure". In incomplete markets, exact replication is impossible and holding an option is a genuinely risky business, meaning that no *preference independent* pricing formula is possible. In technical terms, the problem is that no *unique* martingale measure exists. A variety of approaches have been suggested to get round this problem, none of them perhaps entirely satisfactory. Here we show that if option pricing is imbedded in a utility maximization framework, i.e. the potential option purchaser's attitude to risk is specified, then a unique measure emerges in a very natural way.

An investor with concave utility function U and starting with initial cash endowment x forms a dynamic portfolio whose cash value at time t is $X_x^\pi(t)$ when he uses trading strategy $\pi \in \mathcal{T}$, where $\mathcal{T}$ denotes the set of admissible trading strategies. His objective is to maximize expected utility of wealth at a fixed final time T; we denote

$$V(x) = \sup_{\pi \in \mathcal{T}} E[U(X_x^\pi(T))]. \tag{1}$$

Throughout the paper it will be assumed that the utility function U is non-decreasing and C^2 on R_+ with $U' > 0, \lim_{x \to 0} U'(x) = \infty$ and



$\lim_{x\to\infty} U'(x) = 0$. We ask the question whether the maximum utility in (1) can be increased by the purchase (or short-selling) of a European option whose cash value at time T is some non-negative random variable B, the purchase price at time zero being p. We use a "marginal rate of substitution" argument: p is a fair price for the option if diverting a little of his funds into it at time zero has a neutral effect on the investor's achievable utility. This is an entirely traditional approach to pricing in economics — see [6] for references — but does not appear to have been used much in an option pricing context.

[6] D. Duffie, *Dynamic Asset Pricing Theory*, Princeton University Press, 1992.

To state the definition in precise terms, we need the function W given as

$$W(\delta, x, p) = \sup_{\pi \in T} EU\left(X^{\pi}_{x-\delta}(T) + \frac{\delta}{p}B\right).$$

δ : cash 購入證

δ/p : 購入[illegible]

Definition 1 *Suppose that for each (x, p) the function $\delta \mapsto W(\delta, x, p)$ is differentiable at $\delta = 0$ and there is a unique solution $\hat{p}(x)$ of the equation*

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0.$$

Then $\hat{p}(x)$ is the fair option price at time 0.

This definition will clearly reproduce the Black-Scholes value if perfect hedging is possible. The argument is as follows: suppose p_0 is the perfect-replication value and the option is offered for p. The investor buys δ/p options with cash δ, investing the remaining $x-\delta$ in a portfolio. A moment's thought shows that his optimal procedure is to short the hedging portfolio, whose value is $\delta p_0/p$ and invest his cash fund of $x+\delta(p_0/p-1)$ optimally, attaining an expected utility of $V(x+\delta(p_0/p-1))$. (The option and short hedging fund have equal and opposite value at time T.) The marginal rate of substitution is therefore

$$\frac{d}{d\delta}V\left(x+\delta\left(\frac{p_0}{p}-1\right)\right)\bigg|_{\delta=0}=\left(\frac{p_0}{p}-1\right)V'(x).$$

Evidently, this is equal to zero exactly when $p=p_0$.

In general, if the investor diverts δ into options and uses trading strategy π then his expected utility is

$$E\left[U\left(X^{\pi}_{x-\delta}(T)+\frac{\delta}{p}B\right)\right]=E[U(X^{\pi}_{x-\delta}(T))]+\frac{\delta}{p}E[U'(X^{\pi}_{x-\delta}(T)B)]+o(\delta). \quad (2)$$

We now need the following lemma.

Lemma 2 *Let $f : A \times R \to R$ be a function, where A is some set, and for $\delta \in R$ define*

$$v(\delta) := \sup_{\pi \in A} f(\pi, \delta).$$

Suppose that, for some $\delta_0 \in R$, v is differentiable at δ_0, there exists $\pi^* \in A$ such that $v(\delta_0) = f(\pi^*, \delta_0)$ and the function $\delta \mapsto f(\pi^*, \delta)$ is differentiable at δ_0. Then

$$\frac{d}{d\delta} v(\delta_0) = \frac{\partial}{\partial \delta} f(\pi^*, \delta_0).$$

We can now give a general option pricing formula based on Definition 1.

Theorem 3 *Suppose that V is differentiable at each $x \in R_+$ and that $V'(x) > 0$. Then the fair price $\hat{p}(x)$ of Definition 1 is given by*

$$\hat{p} = \frac{E[U'(X_x^{\pi^*}(T))B]}{V'(x)}. \tag{3}$$

The proof is obtained by evaluating the derivative with respect to δ of the maximum utility at $\delta = 0$, using (2) and Lemma 2, giving a value of

$$-V'(x) + \frac{1}{p} E_x[U'(X_T^{\pi^*} B)],$$

from which (3) follows.

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Pricing weather derivatives by marginal value

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Abstract

Weather derivatives are a classic incomplete market. This paper gives a preliminary exploration of weather derivative pricing using the 'marginal substitution value' or 'shadow price' approach of mathematical economics. Accumulated heating degree days (HDD) and commodity prices are modelled as geometric Brownian motion, leading to explicit expressions for swap rates and option values.

1. Introduction

Many companies are exposed to 'weather risk'. For concreteness, we shall think in terms of an energy company supplying gas to a retail distributor. If a winter month such as January is unusually warm then the company's profits are adversely affected because of the reduced *volume* of gas sold. Note that this is a separate issue from *price risk* which may also be present. The company can partially hedge the volume risk by trading in weather derivatives, which are normally defined as follows (see Geman [4] for extensive background information). Let T_i, the 'temperature on day i' be the average of the maximum and minimum temperatures in degrees Celsius on that day at a specific location (London Heathrow Airport in the UK). The daily number of 'heating degrees' is $\text{HDD}_i = \max(18 - T_i, 0)$ and the accumulated 'heating degree days' (HDDs) over a one-month (31-day) period ending at date t is $X_t = \sum_{i=0}^{30} \text{HDD}_{t-i}$. Over-the-counter contracts are written with X_t as the 'underlying asset'. These may be swaps, the payment at time T being $A(\kappa - X_T)$ where A is the point value and κ a fixed number of accumulated HDDs, or they may be options with exercise value $A \max(X_T - K, 0)$ for a given strike K. The question is, what is the value of these contracts, i.e. the level of the fixed side κ such that the swap has zero value, or the premium to be paid at time 0 for the call option.

Since there is no liquid market in these contracts, Black–Scholes style pricing is inappropriate. Valuation is generally done on an 'expected discounted value' basis, discounting at the riskless rate but under the physical measure, which throws all the weight back onto the problem of weather prediction.

2. Pricing formulae

We model the accumulated HDDs (over, say, a one-month sliding window ending at time t) by a log-normal process X_t

satisfying

$$dX_t = \nu X_t dt + \gamma X_t dw_1(t). \qquad (1)$$

Thus at time T,

$$X_T = \exp(m(T) + \gamma w_1(T)) \qquad (2)$$

where

$$m(T) = \log X_0 + \left(\nu - \tfrac{1}{2}\gamma^2\right) T. \qquad (3)$$

For pricing a weather derivative maturing at time T the main object of concern is simply the one-dimensional random variable X_T, and our basic assumption is that this is log-normal, as indicated by (2). We suppose that the volume of gas sold per unit time is some function $v(t) = v(X_t)$ and suppose that—at least over some range—we can take $v(\cdot)$ as linear: $v(t) = \alpha X_t$. The profit is therefore $Y_t = \alpha X_t S_t$, where S_t is the spot price.

As is conventional, we suppose the price to be log-normal:

$$dS_t = \mu S_t dt + \sigma S_t dw_2(t). \tag{4}$$

In these equations, w_1, w_2 are standard Brownian motions with correlation $E[dw_1 dw_2] = \rho dt$. From (1) and (4), Y_t satisfies

$$dY_t = \theta Y_t dt + \xi Y_t dw(t) \tag{5}$$

with $Y_0 = \alpha S_0 X_0$, where

$$\theta = \nu + \mu + \rho\sigma\gamma$$

and

$$\xi = \sqrt{\gamma^2 + \sigma^2 + 2\rho\gamma\sigma}.$$

The new Brownian motion is

$$dw = \frac{1}{\xi}(\gamma dw_1 + \sigma dw_2).$$

Suppose the weather derivative has exercise value $B(X_T)$ at time T. In [2] we gave a valuation formula for an investor whose overall objective is to maximize the expected utility $E[U(H_T)]$ of his portfolio value H_T at time T. This value is

$$\hat{p} = \frac{E[U'(H_T^*)B(X_T)]}{V'(\eta)} \tag{6}$$

where H_T^* is an optimal portfolio of tradeable assets with initial endowment η and $V(\eta) = E[U(H_T^*)]$. In the present case our producer has no investment decisions: he simply produces up to the level of current demand and sells at market price. Thus $H_T^* = Y_T$, the profit at time T. We will assume utility is logarithmic, $U(y) = \log y$, and then it is easy to see that $V(y) = \log y + \text{const}$. Thus $V'(y) = 1/y$ and the pricing formula (6) becomes

$$\hat{p} = E\left[\frac{Y_0}{Y_T} B(X_T)\right] \tag{7}$$

Proposition 1. *The zero-cost swap rate at time 0 is*

$$\hat{\kappa} = e^{(\nu - \gamma^2 - \rho\sigma\gamma)T} X_0. \tag{8}$$

The option value (4) with $B(x) = [x - K]^+$ is given by

$$\hat{p} = \mathrm{BS}(x_0, K, r, q, \gamma, T), \tag{9}$$

the Black–Scholes call-option formula, in which the 'riskless rate' r and 'dividend yield' q are given by

$$r = \mu + \nu - \gamma^2 - \sigma^2 - \rho\sigma\gamma, \tag{10}$$

$$q = \mu - \sigma^2. \tag{11}$$

Proof. Defining $Z_t = Y_0/Y_t$ we find using (5) and the Ito formula that

$$dZ_t = -rZ_t dt - \xi Z_t dw_t, \qquad Z_0 = 1,$$

where r is given by (10). Thus

$$\begin{aligned} \hat{p} &= E[e^{-rT} \exp(-\xi^2 T/2 - \xi w_T) B(X_T)] \\ &= \hat{E}[e^{-rT} B(X_T)] \end{aligned} \tag{12}$$

where $\hat{E}$ denotes expectation with respect to the measure $\hat{P}$ defined by

$$\frac{d\hat{P}}{dP} = \exp(-\xi^2 T/2 - \xi w_T).$$

← ギルサノフ変換を行う

Recall that X_t satisfies (1). We find that $E[\mathrm{d}w\mathrm{d}w_1] = \rho_1\mathrm{d}t$ where $\rho_1 = (\gamma + \rho\sigma)/\xi$, and $\mathrm{d}\hat{w} = \mathrm{d}w + \xi\mathrm{d}t$ is Brownian motion under $\hat{P}$ by the Girsanov theorem. It follows that under $\hat{P}$ there is a Brownian motion $\hat{w}_1$ such that

$$\mathrm{d}X_t = (\nu - \rho_1\gamma\xi)X_t\mathrm{d}t + \gamma X_t\mathrm{d}\hat{w}_1(t). \qquad (13)$$

We note that the 'drift' is $\nu - \rho_1\gamma\xi = \nu - \gamma^2 - \rho\sigma\gamma = r - q$ with q defined by (11). Thus when $B(X_T) = X_T - \kappa$ we have $\hat{p} = \mathrm{e}^{-qT}X_0 - \mathrm{e}^{-rT}\kappa$, so the zero-cost swap rate is $\hat{\kappa} = \mathrm{e}^{(r-q)T}X_0$; this is (8). In the case of a call option, $B(X_T) = [X_T - K]^+$, and the result (9) follows from (12) and (13). □

2.1. Comments

- The swap rate $\hat{\kappa}$ is not equal to the physical measure forward HDD $e^{\nu T} X_0$ but is equal to $e^{\tilde{r}T} X_0$ where $\tilde{r} = \nu - \gamma^2 - \rho\sigma\gamma$ depends on both HDD and price volatility.
- If the price is constant ($\mu = \sigma = 0$) then the 'dividend yield' q of (11) is zero and the option price (9) is just the no-dividend Black–Scholes price with 'riskless rate' $\tilde{r}$. Note that q depends only on the *price* parameters. For general μ, σ the discount rate is $r = \tilde{r} + \mu - \sigma^2$. The effect of price volatility on option value is explored in section 4 below.
- The pricing formulae do not involve the demand sensitivity α, so it is unnecessary to estimate this parameter. Since $Y_t = \alpha X_t S_t$, adjusting α is equivalent to changing the units of the price process S_t. The pricing formula is invariant under such changes; it only depends on the drift and volatility parameters of S_t.
- The riskless rate of interest does not come into the picture in view of the absence of any trading involving the riskless asset.

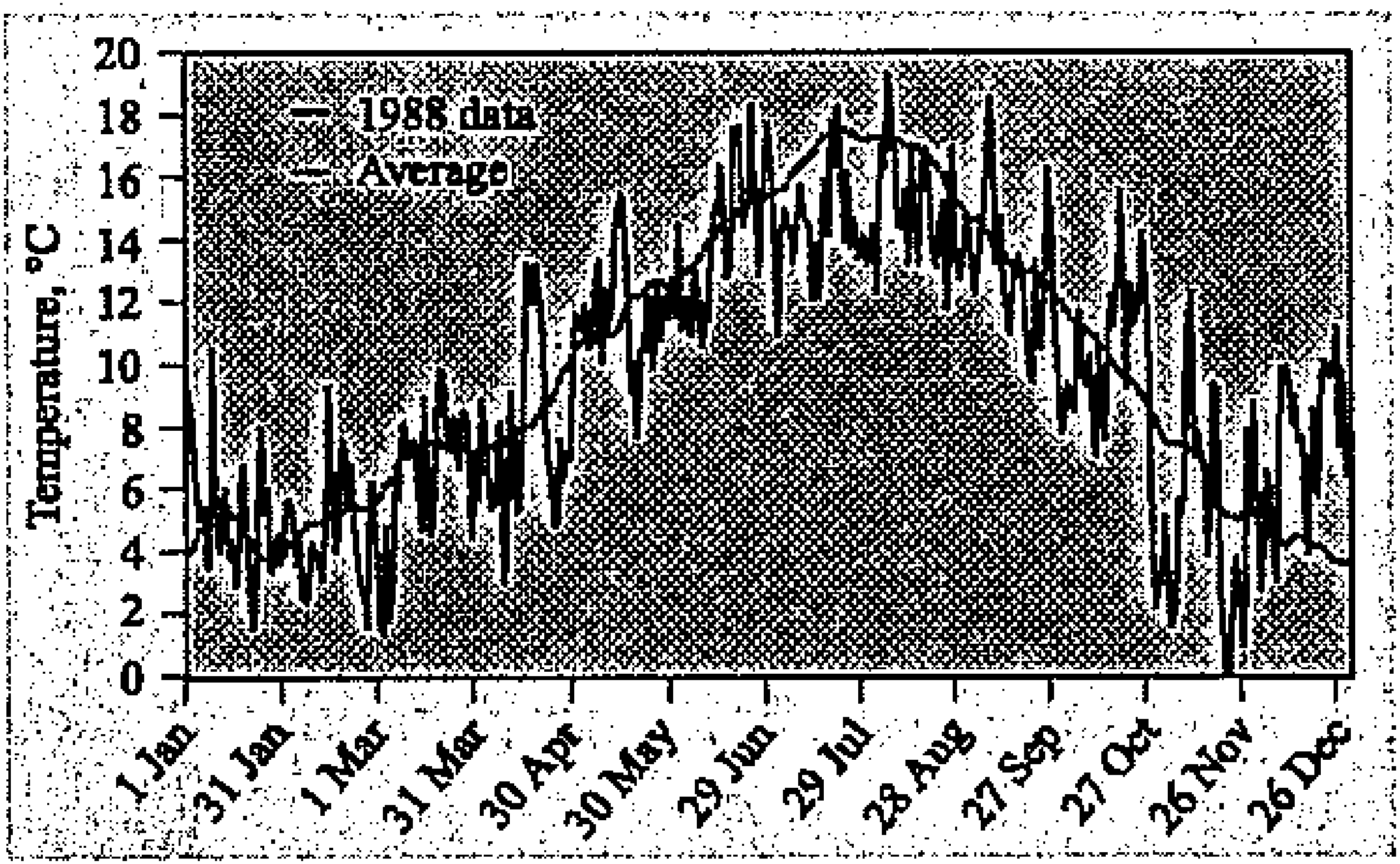


Figure 1. Long-term average temperature and temperatures for 1988.

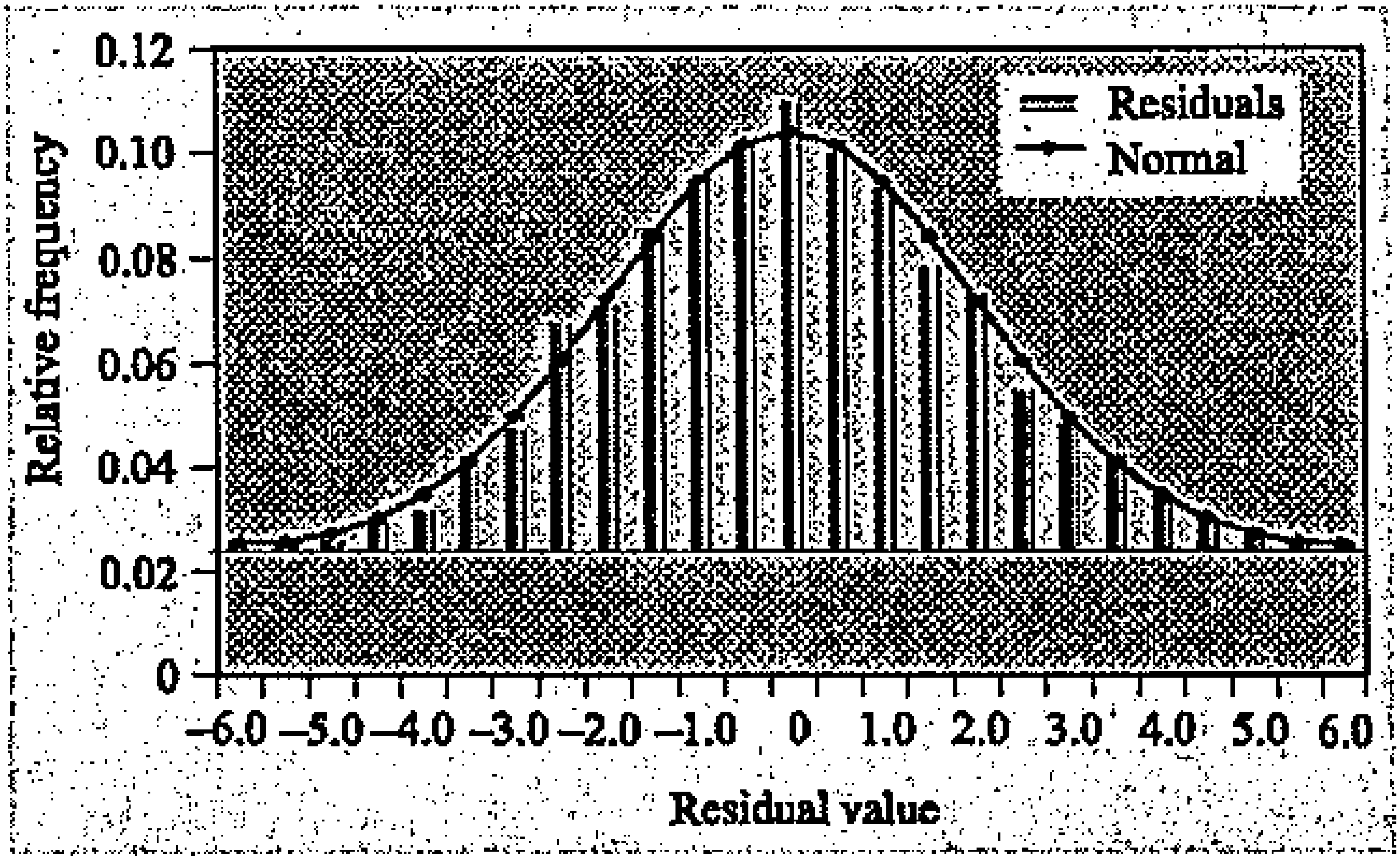


Figure 2. Empirical distribution of residuals and best normal fit.

3. HDD modelling

Weather prediction is a big subject. Nevertheless, some simple things can be said that provide an adequate basis for at least some derivative pricing problems. The objective of this section is to provide just enough evidence to convince the reader that a log-normal model for accumulated HDDs is not at all unreasonable, and to give easily-implemented parameter estimation methods. We do not claim to be providing an exhaustive analysis of the data.

The data set[2] consists of daily temperatures (average of maximum and minimum) at Birmingham, England for the 11 year period 1988–1998. We denote this series by $\{T_i, i = 1, \ldots, 4015\}$, while $\{\overline{T}_i, i = 1, \ldots, 4015\}$ denotes the long-term average temperature. For each i, $\overline{T}_i$ is obtained by taking the average of the 11 temperatures on the corresponding date and then smoothing the series by moving-average smoothing. Thus the $\overline{T}_i$ series is periodic. Figure 1 shows the two series for the year 1988.

As many researchers have noted, the deviation $D_i = T_i - \overline{T}_i$ is accurately modelled as a low-order autoregression (AR):

$$D_i = \sum_{k=1}^{n} a_k D_{i-k} + b\epsilon_i, \tag{14}$$

where ϵ_i is a sequence of independent unit-variance Gaussian residuals. Here we restrict ourselves to the first-order case $n = 1$. The least-squares estimates $\hat{a}, \hat{b}$ of the parameters a_1, b based on the whole data set are $\hat{a} = 0.70$, $\hat{b} = 1.99$. These estimates are quite stable when estimated over, say, three-year windows of data. A more sophisticated analysis would allow for seasonally-dependent variability b, but we have stuck to a constant-parameter model. The residual sequence is then $\hat{\epsilon}_i = (D_i - \hat{a} D_{i-1})/\hat{b}$. The first ten estimated correlation coefficients—again based on all the data—of the residuals are all in the range ± 0.045, indicating that the residuals are reasonably 'white'. What is more striking is the residual empirical distribution, shown in figure 2 along with the normal density with the same mean and variance. The fit is astonishingly good. No financial time series behaves like this!

We are thus happy to represent the deviation from long-run average temperature as a Gaussian first-order AR.

The AR (14) with $n = 1$ and $|a_1| < 1$ converges to a stationary distribution with mean zero and standard deviation $\Sigma = b/\sqrt{1 - a_1^2}$. The correlation coefficient at lag k is a_1^k. Since $0.7^{15} = 0.0047$ we see that the deviations from long-run average at any time more than two weeks ahead are essentially independent of today's value. Thus if we want to estimate the distribution of accumulated HDDs over a one-month period starting at any time more than two weeks ahead we can simulate D_i from the stationary distribution and take the simulated temperature as $T_i = \overline{T}_i + D_i$. Figure 3 shows the empirical distribution and best log-normal fit for accumulated HDDs over the month of May, using the estimated parameters $\hat{a}, \hat{b}$. The fit is excellent, and similar results are obtained for other months. In fact, this is not surprising: the mean temperature in May is around 11 degrees and with the estimated parameters $\Sigma = 2.77$. Thus the 18 degree barrier is 2.5 standard deviations away from the mean, so that the accumulated HDD is close to being normally distributed. The log-normal distribution with the same mean and variance gives an excellent approximation to standard option values, although of course the tail behaviour is radically different.

4. Example

As an example, consider a call option on the accumulated HDDs for May 2001, written on 1 November 2000 with strike $K = 560$. From our simulations, we know that the mean and standard deviation of the accumulated May HDDs are 577 and 35 respectively. (The option is thus 'at the money'.) Referring to the representation (2), we find by calculating the mean and variance that $\gamma = 8.82\%$ and $m(T) = 6.325$. If we take $X_0 = 560$ then this implies $\nu = -0.13\%$. For the price process we take $\mu = 0$, so there is no drift in the price. However, as can be seen in figure 4, the value of the option depends significantly on the *price* volatility. Under the measure $\hat{P}$, X_t has drift $r - q = \nu + \gamma^2 - \rho\sigma\gamma$, while the discount factor is $r = (r-q)+\mu-\sigma^2$. If $\rho = 0$, the drift is independent of σ and the option value increases with σ because the discount factor is reduced. For $\rho > 0$ both drift and discount factor are reduced with increasing σ; the net effect is decreasing option values except for very small ρ, as the chart shows. When $\rho < 0$ the effects are in the same direction: less discounting and higher drift lead to increasing value.

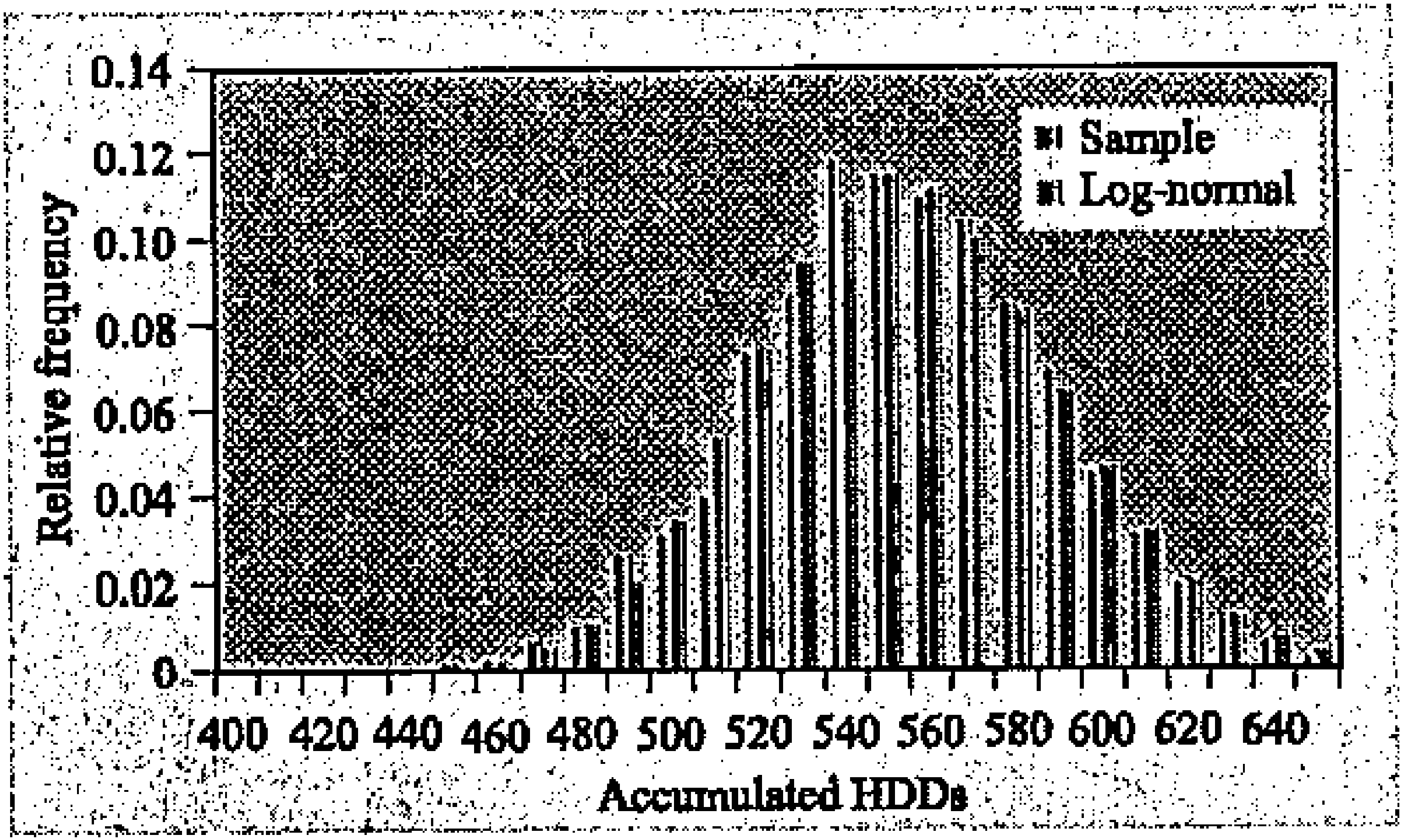


Figure 3. Simulated HDD distribution and best log-normal fit.

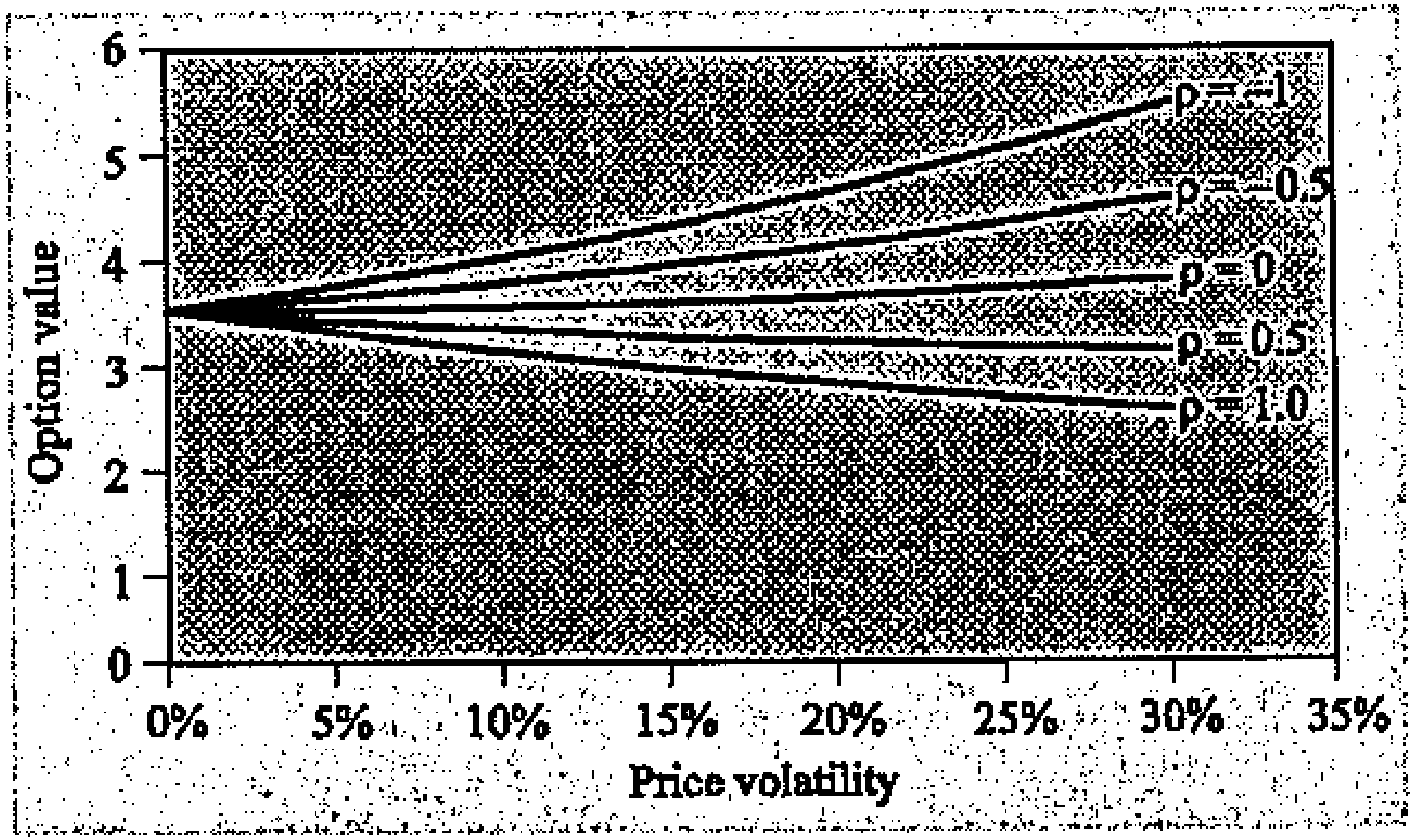


Figure 4. Option value as a function of price volatility, for different values of correlation.

References.

R.Miura (3). "Edokko Options: A New Framework of Barrier Options" (with Takahiko Fujita), Asia-Pacific Financial Markets, Vol.9.2, 2002

R.Miura (4). “Rank Process, Stochastic corridor and Applications to Finance.” 529-542. Chapter 26. Advances in Statistical Modeling and Inference: Essays in Honor of Kjell Doksum. World Scientific.2007.

Dear Students in this class,

pdf file (scanned copy) of these reference papers are available.

If you would like to read them, please let me know by e-mail.

Ryozo Miura

Homework

: (1). Derive the pricing function of the standard (usual) Barrior option: perform the calculation.

: (2). Discuss if Brownian quantile is a Markov process.

: Please submit, by e-mail, your homework by Monday noon (May 2) so that I can get them printed.

Appendix

Incomplete attempt for weather derivatives and a firm value with probability of default.

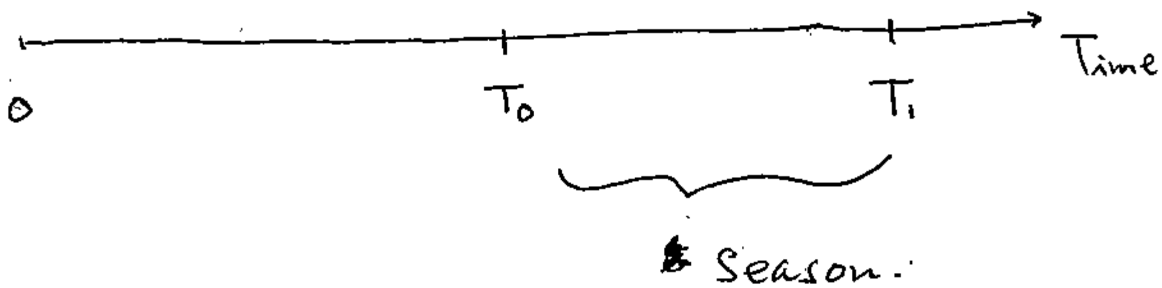
An attempt (unpublished work: incomplete)

Interested in

: how much the weather derivative helps the value of a firm.

: how many contracts a firm should buy in order to decrease a probability of default?

1. 企業価値と天候への依存性の表現



Season ① 夏場　暑さにより、飲料品の売上が変動する。

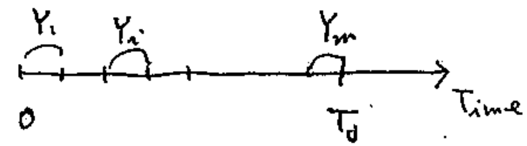
② 冬場　降雪量に依り、スキー場のにぎわいが変動する。

1. 企業価値 V_t, $t \in [0, T_1]$

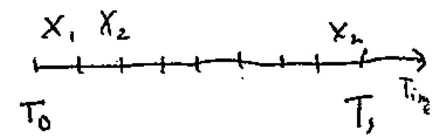
0時点からT_0時点までは、天候デリバティブを必要としない時期とする。この期間中は、通常の営業活動により、企業価値が推移する、とする。

$$V_{T_0} = V_0 \cdot e^{\mu_0 (T_0 - 0) + \sigma_0 (W_{T_0} - W_0)}$$

$$\doteq V_0 \cdot e^{Y_1 + Y_2 + \cdots + Y_m}$$



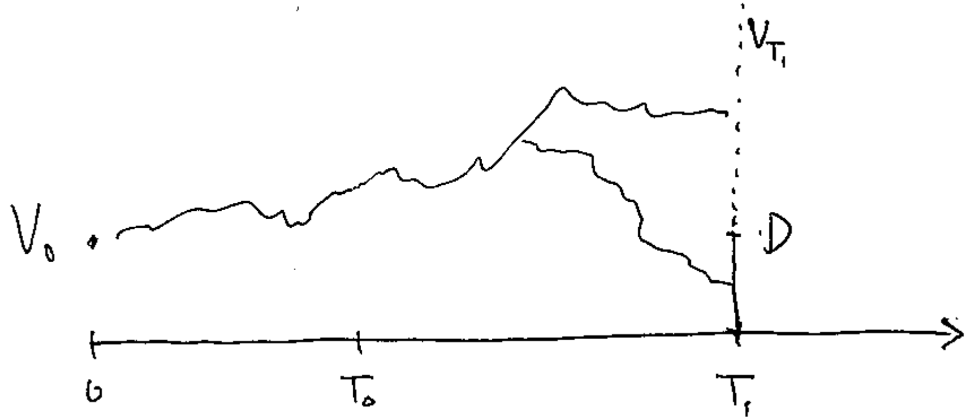
期間$[T_0, T_1]$をn日として、日々の投資収益率を$X_1, \cdots, X_n$と書く。つまり、



$$V_{T_1} = V_{T_0} \cdot e^{X_1 + X_2 + \cdots + X_n}.$$

$X_1, X_2, \cdots, X_n$は、天候に依存した確率分布をもつとする

倒産リスクについて、この枠組の中で表現しておく。



$V_{T_1} < D$ である場合をデフォルトと定義する

デフォルト確率 $= P\{V_{T_1} < D\}$

この確率を0時点で計量すると

$$P\{V_{T_1} < D \mid V_0\} = P\left\{\log\frac{V_{T_1}}{V_0} < \log\frac{D}{V_0}\right\}$$

$$= P\left\{\begin{pmatrix}\mu_0(T_0-0) + \sigma_0 W_{T_0} \\ + X_1 + \cdots + X_n\end{pmatrix} < \log\frac{D}{V_0}\right\}$$

$$V_t = e_t - p_t \cdot \xi_{T,t}$$

$$V_T = e_T + X_{T,t} \cdot \xi_{T,t}$$

解法

$$\Pi_t(e_t, \xi) = u(e_t - p_t \cdot \xi) + E_t\left[\beta \cdot u(e_T + X_{T,t} \cdot \xi)\right]$$

$$\frac{\partial}{\partial \xi}\Pi_t(e_t, \xi) = -p_t \cdot u'(e_t - p_t \cdot \xi) + E_t\left[X_{T,t} \cdot \beta \cdot u'(e_T + X_{T,t} \cdot \xi)\right]$$

$$\left.\frac{\partial}{\partial \xi}\Pi_t(e_t, \xi)\right|_{\xi=0} = -p_t \cdot u'(e_t) + E_t\left[X_{T,t} \cdot \beta \cdot u'(e_T)\right]$$

これをゼロに等しいと置いて、 $p_t = \dfrac{1}{u'(e_t)} E_t\left[X_{T,t} \cdot \beta \cdot u'(e_T)\right]$

~~$u'(e_t)$ は e_t~~

(注.) Davis論文「Option Pricing in Incomplete markets」の第1節と同等である。

研究メモ1

↙ ($V_T + \xi \cdot X_T$) と考えるべく…

$u\left(V_T + \frac{\xi}{p} X_T\right)$ としても結果は同じ

$u(\cdot)$ の中に $V_T < D$ をきらう要素が十分

$$\Pi_t = u(V_t - \xi \cdot p_t) + E_t\left[\beta \cdot u\left(V_T \cdot e^{\xi \cdot X_T}\right)\right] + \ell\left(V_T^H > D\right)$$

$$\frac{\partial}{\partial \xi}\Pi_t = -p_t \cdot u'(V_t - \xi \cdot p_t) + E_t\left[\beta \cdot V_T \cdot e^{\xi \cdot X_T} \cdot X_T \cdot u'\left(V_T \cdot e^{\xi X_T}\right)\right]$$

$$\left.\frac{\partial}{\partial \xi}\Pi_t\right|_{\xi=0} = -p_t\, u'(V_t) + E_t\left[\beta \cdot V_T \cdot X_T \cdot u'(V_T \cdot 1)\right]$$

$$\left.\frac{\partial}{\partial \xi}\Pi_t\right|_{\xi=0} = 0 \Rightarrow p_t = \frac{1}{u'(V_t)} E_t\left[\beta \cdot V_T \cdot X_T \cdot u'(V_T)\right]$$

$$= E_t\left[\beta \frac{u'(V_T)}{u'(V_t)} \cdot V_T \cdot X_T\right]$$

$u(v) = \log v$ の場合 $u'(v) = \frac{1}{v}$ $\Rightarrow E_t[V_t \cdot \beta \cdot X_T]$ に等しい。

V_T が入っているので企業価値理論が表現に入る。

—。—

V_T について。

$(A_1 + \cdots + A_n) + (\alpha(H_1)B_1 + \cdots + \alpha(H_n)B_n)$

…を説明に入る.

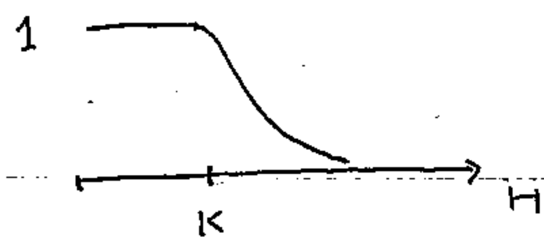
V_T について.

$$V_T = V_t\, e^{(A_1+\cdots+A_n)+(\alpha(H_1)B_1+\cdots+\alpha(H_n)B_n)}$$

A_i は天候の影響をうけない部分とする.

H_i は天候指標とし、$\alpha(H_i)$ は H_i による影響,

B_i は $\alpha(H)=1$ つまり天候リスク0のときの営業利益率.



$\alpha(H)<1$ では天候に依り利益率が下がる度合 $(1-\alpha(H))$ を示す.

天候デリバティブによるヘッジ

T時点でペイオフにより単位デリバティブ当り利益率 $\underbrace{\sum_{i=1}^{n}(1-\alpha(H_i))B_i}_{=X_T}$ だけ回復されるようにデリバティブを作る

$$V_T\cdot e^{\xi X_T} = V_T\cdot e^{\xi\cdot\sum_i(1-\alpha(H_i))B_i}$$

$=$

こちらの方が理屈合う(はりがね氏)

① $V_T = (V_t - \xi p_t) e^{Y_{T-t}}$

又は. この場合右ページの議論は少し変わる

② ξp_t は借り入れ資金により調達するとして T時点にて負債額に含める.

もともとの負債を D_0 とし. T時点における返済額

$D_T = D_0 + \xi p_t \cdot e^{r(T-t)}$

とする.

倒産確率.

$P\{ V_T e^{\xi X_T} < D_T \}$

$\xi = 0$ の場合 (デリバティブをもたない場合)

$P\{ V_T < D_0 \}$

ξ単位の天候デリバティブをもつ場合

X_T はデリバティブ一単位分のペイオフ.

$P\{ V_T e^{\xi X_T} < D_0 + \xi p_t \cdot e^{r(T-t)} \}$

(I) 2002.8.12 のメモ

c は任意として p_c を定め

c を倒産確率最小化により定める

ξ と c と見立てて右の倒産確率が c の関数としてどのように形られるかをみると最小化ともとて面白い。オプションのギリから c の任か p_c が一定という性質をもつことが証明

もって

X_T はデジタル・バイナリー単位分のペイオフ。

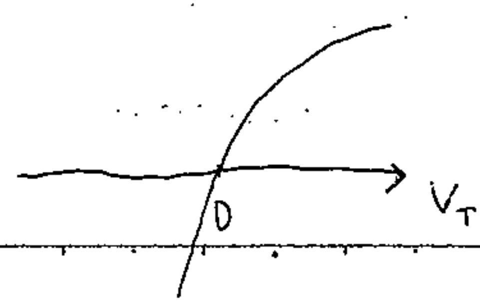
$\rightarrow P\{V_T e^{\xi X_T} < D_0 + \xi p_c e^{r(T-t)}\}$

議論のポイント。

倒産確率を小さくしたいというのは（経営者の）効用でもある。従って効用関数 $u(\cdot)$ にそのリスク回避姿勢が反映されていないといけない。T時点での企業資産が負債を下回ると効用がぐっと負の方向へ伸びる関数形でなくては表わされない

$u\left(\frac{V_T}{D_T}\right) = \left(\frac{V_T}{D_T}\right)^{\gamma} - 1$ 導入

$\left(\frac{V_T}{D_T}\right) < 1$



研究メモ 2

ペイオフの中に用いられる乗数 c を意味づける。

$$V_T = V_t \, e^{\sum A_i + \sum \alpha(H_i) B_i}$$

$$X_T = c \cdot \sum_{i=1}^{n} \max\{K - H_i, 0\} = c \sum (K - H_i) I\{K > H_i\}$$

又は

$$X_T = c' \left(\sum_{i=1}^{n} I\{K > H_i\} - 3 \right)$$

02.9.6

のように置くと、

$$p_t^c = E_t\left[\beta \cdot \frac{u'(V_T)}{u'(V_t)} \cdot V_T \cdot X_T\right] \qquad \left(p_t^c = c \cdot p_t^1\right)$$

$$= E_t\left[\beta \cdot \frac{u'(V_T)}{u'(V_t)} \cdot V_T \cdot c \sum_{i=1}^{n} \max\{K - H_i, 0\}\right]$$

これらの収益率の[illegible]しか[illegible]

又は

$$= E_t\left[\beta \cdot \frac{u'(V_T)}{u'(V_t)} \cdot V_T \cdot c' \cdot \left(\sum_{i=1}^{n} I\{K > H_i\} - 3 \right)\right]$$

→ 倒産確率とcの関係をみよ

[X_T (ペイオフ)の内容に応じて c, c' が定まる。これは一般論

ここでは c の大きさを決定する枠組みがない → 経営者が何回リスクをとったか

論文 のように書くことができ、これが要点

c, c' の定め方が見えてくる。つまり 企業理論を考慮しないと

[illegible]

これから決まる

c, c' の定め方が見えてくる。つまり 企業理論を考慮しないで

$V_t = a$ の企業理論に対して
$C = 1$ が適切であれば
しかし ?

とすると $\frac{V_t}{a} = c$
のように定まる
しかし
適切
とするのか?

$$p_t = E_t\left[\beta \cdot \frac{u'(\ )}{u'(\ )}\ c \cdot \Sigma \max\{K - H_i, 0\}\right]$$

$$= c\ E_t\left[\beta \cdot \frac{u'(\ )}{u'(\ )}\ \Sigma \max\{K - H_i, 0\}\right]$$

に対応させて

$$p_t = E_t\left[\beta \cdot \frac{u'(\ )}{u'(\ )}\ V_T\ \Sigma \max\{K - H_i, 0\}\right]$$

これは信用リスク
割引とリスク回避度
に依存しているのでは
ないか?

これを例えば V_T の期待値(定数)のようなものに差しかえて $E_t[\cdot]$ の外へ出すと c が得られる

注意
$u(y)=\log y$
とおいて
$u'(y)=\frac{1}{y}$
なので
V_T
が消えて

$P_t = E_t[\beta \cdot V_t \cdot c \cdot \Sigma \max\{K-H_i, 0\}]$

2002.8.12 メモ

$\frac{u'(V_T)}{u'(V_t)}$ は企業現在価値に依存しない量

$V_T = V_t \cdot e^{\square}$ は企業現在価値 V_t と成長率

β

$P_t = c \cdot E_t\left[\beta \cdot \frac{u'(V_T)}{u'(V_t)} \cdot V_t \cdot e^{\square} \cdot \Sigma \max\{K-H_i, 0\}\right]$

となり 現在価値の価格の c倍になるだけー ← それでは cの決定は どうやって行うか が問題となる

2002.8.13

2002.4.6.

Cの値と価格弾力性eの関係をみるためには

V_Tも天候の関数にする。

X_Tなしの場合のV_Tの分布とX_T付きのV_Tの分布を比べる。

（Cが入っている。）

Cの大きさに応じて価格弾力性は下がるか。（Cが大きいと購入され支払う天候デリバの代金も考慮しないといけない）

Remarkすること

Cについて

$$E_t\left[\beta\cdot\frac{u'(V_T)}{u'(V_t)}\cdot V_T\cdot\sum_{i=1}^{n}\max\{K-H_i,0\}\right]$$

$$\Leftrightarrow\ E_t\left[\beta\cdot\frac{u'(V_T)}{u'(V_t)}\cdot C\cdot\sum_{i=1}^{n}\max\{K-H_i,0\}\right]$$

$\exists$

$$\Rightarrow\ C=\frac{E_t\left[\beta\cdot\frac{u'(V_T)}{u'(V_t)}\cdot V_T\cdot\sum_{i=1}^{n}\max\{K-H_i,0\}\right]}{E_t\left[\beta\cdot\frac{u'(V_T)}{u'(V_t)}\cdot\sum\max\{K-H_i,0\}\right]}$$

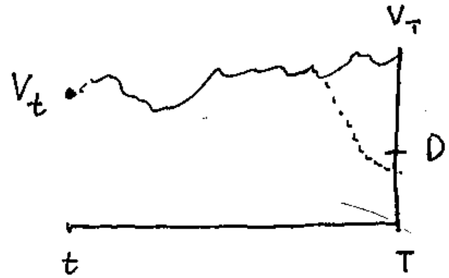
Davis論文では [P.〇〇 定理5のことだろう]

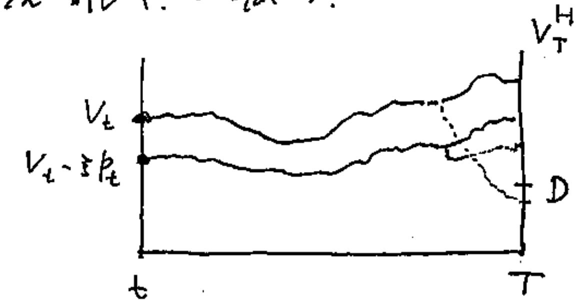
$\frac{u'(V_T)}{u'(V_t)}$ の替りに $e^{-\delta\tau}$ にしている。それにより計算が簡単になる

この指数にはロジック(経済学的な)をおいていないらしい (中村)

よく見ると utility maximization から導かれるといっている

天候デリバティブによるヘッジと企業倒産確率の減少





$$V_T = V_t \, e^{\int_t^T (\cdots)}$$

天候デリバを
ξ単位買うと
D は $D + \xi p_t e^{r(T-t)}$
に変る

$$V_T^H = V_T^* \cdot e^{\xi X_T} = (V_t - \xi \cdot p_t) \cdot e^{\int_t^T (\cdots)} \cdot e^{\xi X_T}.$$

$$P\{V_T < D\} = P\{V_t \cdot e^{\int_t^T (\cdots)} < D\} = P\left\{\int_t^T (\cdots) < \log \frac{D}{V_t}\right\}$$

$$P\{V_T^H < D\} = P\{(V_t - \xi p_t) e^{\int_t^T (\cdots)} \cdot e^{\xi X_T} < D\}$$

$$= P\left\{e^{\int_t^T (\cdots)} e^{\xi X_T} < \frac{D}{V_t - \xi p_t}\right\}$$

$$= P\left\{\int_t^T (\cdots) + \xi X_T < \log \frac{D}{V_t - \xi p_t}\right\}$$

ヘッジ要素，　天候デリバティブ購入単位数量に依存して倒産確率が変わる。

天候デリバティブ1単位が"何であるか"をきちんと示す

デリバティブのデザイン。

本来は企業財務、企業価値の中に投資収益率として組み込んで考えることが理論的にはよい。しかし商品としては(天候条件の関数として)金額として表示されている。金額の方が商品として分かりやすいからそのようになっているのだろう。その金額が企業財務にどのような意味をもつのかを理論上で、みえさせておくとよい。