Title.
"Some Ideas and Results in Probability and Statistics for Quantitative Finance".

Ryozo Miura

(Hitotsubashi University).

Charles University. April 20, 2011.

[1] 1st Day. April 20 (= 3 hours).

["Nonparametric Statistics" for Brownian Motions.]

Definition and Probability Distribution of
: (i). Occupation Time (Empirical Distribution Function),
: (ii).Brownian Quantiles or alpha-quantiles (Order Statistics),
: (iii). Ranks (Rank Statistics).

The probability distribution of these quantities will be derived under the assumptions that the underlying stochastic process is Geometric Brownian Motion.

These could be regarded as a continuous time version of probability theory of cumulative sum of iid random variables.

Using the followings;
Arcsine Law,
First Hitting Time,
Moment Generating Functions,
Feynman-Kac Theorem,
Laplace Transformations.

References.

R.Miura (1) "A Note on Look-Back Options Based on Order Statistics", Hitotsubashi Journal of Commerce and Management. Vol. 27, No.1, November 1992.

(This paper should be regarded as a brief writing to present the ideas. The rigorous works can be found in the papers by Akahori, Dassios and Yor.)

R.Miura (2)."The Distribution of Continuous Time Rank Processes" (with Takahiko Fujita), Mathematical Economics, Vol. 9, 2006

Also first half of Miura(4) below.

There is a website showing all kinds of derivatives where alpha-quantiles options can be found ;

http://www.global-derivatives.com/index.php/options-database-topmenu/13-options-database/16-alpha-quantile-options

A Brief History of α-Quantiles and Ranks

α-quantiles

: Definitions of α-quantiles = Miura(1992)

: Rigorous Treatments for probability distributions = Akahori(1995)

: Some properties = Dassios (1995), Yor(1995),Embrechts et'al (1995).

:Further Developments = Dassios (2005) and others(Levy process)

= Fujita et'al (2002,2010)

: (1). J.G.Wendel(1960) 。"Order Statistics of Partial Sums." Annals of Mathematical Statistics.1034-1044 Theorem6.1 (Laplace Transform of the pair of random variables.

＝＝I did not know of this paper when I published the paper (1992) on α-quantiles.

Ranks

Definitions & Probability Distribution =Fujita and Miura (2006)

Forward Starting Stochastic Corridor = Miura (2007)

Our published paper on Hedge Fund Return

: [1] Miura R. , D. Yokouchi and Y. Aoki (2009). "A Note on Statistical Models for Individual Hedge Fund Returns." Math.Meth. Oper. Res. 69. 553-577.

Main References for α-quantiles and Ranks

α-quantiles

:[1]. Akahori, J. (1995). “Some formulae for a new type of path-dependent option.” Ann. Appl. Probab. 5. 383-388.

:[2]. A.N. Borodin and P. Salminen. (2002). Handbook of Brownian Motion - Facts and Formulae, 2nd. edition p.256. (the first edition was published in1996) Birkhauser.

:[3]. Dassios, A. (1995). “The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. Ann. Appl. Probab. 5. 389-398.

:[4]. Dassios,A.(2005). “On the quantiles of Brownian motion and their hitting times.” Bernoulli 11(1), 29–36.

:[5]. Embrecht, P., Rogers, L.C.G. and Yor, M. (1995). “A proof of Dassios’s representation of the α-quantile of Brownian motion with drift.” Annals of Applied Probability. 5,757-767.

:[6]. Fujita, T.(1997). ”On the price of α-percentile options.” Working Paper Series #24, Faculty of Commerce Hitotsubashi University.

:[7]. Fujita, T. (2000). “A note on the joint distribution ofα,βpercentiles and its applications to the option pricing.” Asia-Pacific Financial Markets. Vol.7(4), 339-344.

:[8]. Fujita, T. and Miura, R.(2002). “Edokko Options: A New Framework of Barrior Options.” Asia-Pacific Financial Markets. Vol.9(2),December, 141-151.

: [9]. Fujita, T. and Ishizaka, M.(2002). “An application of New Barrier Options (Edokko Options) for Pricing Bonds with Credit Risk”. Hitotsubashi Journal of Commerce and Management, Vol.37(1) pp.17-23.

:[10]. Miura, R. (1992). “ A note on look-back option based on order statistics.” Hitotsubashi Journal of Commerce and Management. 27,15-28.

:[11]. Yor, M.(1995).”The distribution of Brownian quantiles.” Journal of Applied Probability. 2, 405-416.

Some More References(continued)

Other papers :

Kwok, Fusai, Ballotta, Linestky, and several other people have also worked on derivatives based onα-quantile of Brownian motion with drift.

Rank Processes

:[1]. Fujita, T. and Miura, R. (2006). " The distribution of Continuous Time Rank Process." Advances in Mathematical Economics,Vol.9, 25-32. Springer Verlag.

: [2] Miura, R. (2007). "Rank Processes, Stochastic Corridor and Applications to Finance." Advances in Statistical Modeling and Inference

(Festschrift s for Professor Kjell Doksum, his 65 th. Birthday).

Edited by Vijay Nair. pp.529-542. World Scientific.

:[3]. Kamimura & Miura (2006). "Numerical Aspects of α-quantiles and Ranks." Presented at the 4 th.World Congress of Bachelier Finance Society.

New

Ishii and Fujita (2010 APFM, published online 2009)

"Valuation of a Repriceable Executive Stock Option"

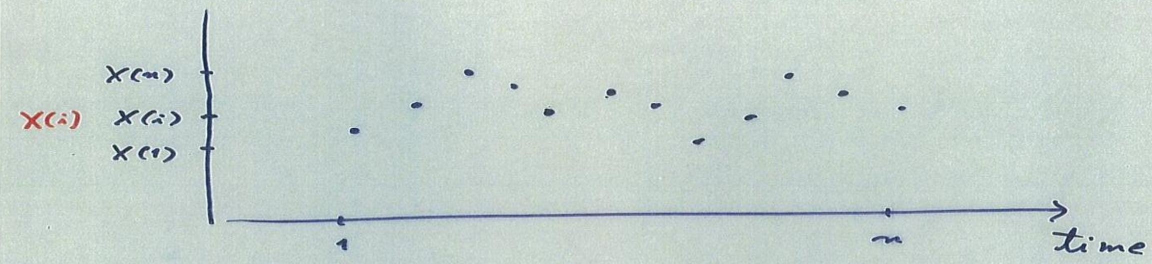
Section 1.
Definitions and Probability Distributions

“Empirical Distribution Functions” :Fixed Level Corridors

“Order Statistics” :Brownian Quantiles

“Ranks” :Stochastic Corridors

To Summarize the Definition & Idea

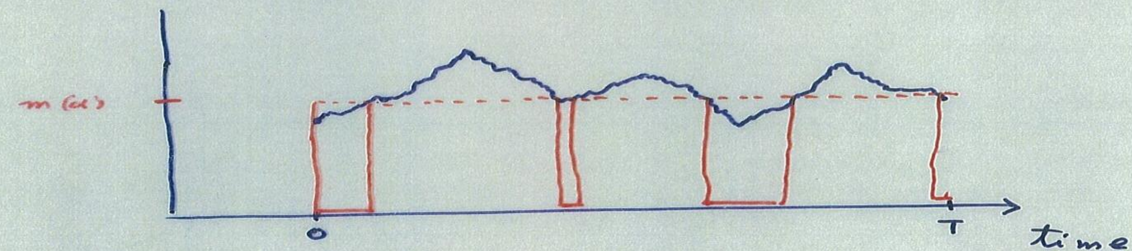


$S(u) \leqq m(\alpha)$ du

$0<\alpha<1$: $m(\alpha)$: $\alpha = \frac{1}{T}\int_0^T I\{X(s) \leqq m(\alpha)\}\,ds$

$S(u) \leqq S(t)$ du

$0<t<T$: Rank of $X(t)$: $R(t) = \frac{1}{T}\int_0^T I\{X(s) \leqq X(t)\}\,ds$



$m(\alpha) < S(t)$

Note: $\Pr\{R(t) > \alpha\} = \Pr\{m(\alpha) < X(t)\}$

:1. “Non-Parametric” Statistics and Exotics.

Three “Non-Parametric Statistics” of a Brownian Motion are:
“Empirical Distribution Functions”,
α-quantiles , and
Ranks

Assume : the Stock price follows a Geometric Brownian Motion:

$$S_u = S_0 e^{X_u} = S_0 e^{\mu u + \sigma W_u} \quad \text{, for u} \in \text{[o,T]}$$

: (1-1) ”Empirical Distribution Function”

Fixed Corridor : Corridor with a Fixed Level K, 0<K

$$F(K) = \frac{1}{T} \int_0^T I\{S_u \le K\} du$$

:(1-2) α-quantiles : m(α,S) . "OrderStatistics"

: S in order to indicate the stochastic process on which the quantiles are defined

For any given $\alpha \in [0, 1]$, α-quantile is defined as a quantity $m(\alpha)$ such that

$$\alpha = \frac{1}{T}\int_0^T I\{S_u \le m(\alpha, S)\}du$$

Note: $m(\alpha, S) = S_0 \exp\{ m(\alpha, X) \}$ where $m(\alpha, X)$ is such that

$$\alpha = \frac{1}{T}\int_0^T I\{X_u \le m(\alpha, X)\}du$$

or more precisely, (now, this is called Brownian quantiles)

$$m(\alpha, X) = \inf\left\{x : \alpha \le \frac{1}{T}\int_0^T I\{X_u \le x\}\mathrm{du}\right\}$$

: (1-3) Ranks

"Rank Statistics" Process = Stochastic Corridor

The rank process is defined by

$$R(t,S) = \frac{1}{T}\int_0^T I\{S_u \le S_t\}du$$

for any prefixed time point t in [0,T].

Note: The Rank does not depend on the initial value S_0.

Also, note: Rank is invariant under shift and scale changes.

$$I\{S_u \le S_t\} = I\{S_0 e^{X_u} \le S_0 e^{X_t}\} = I\{X_u \le X_t\}$$

So we see that R(t,S) = R(t,X).

Also an important relation to α -quantiles is,
Identity of events: $\{m(\alpha, S, [0, T]) < S_t\} = \{R(t,S,[0,T]) > \alpha\}$.

$$\{\frac{1}{T}\int_0^T I\{S_u \le y\}du < \alpha\} = \{m(\alpha) > y\}$$

: Note: These "statistics" are counting how much time the stock prices are under K, St, and m(α), respectively
:***

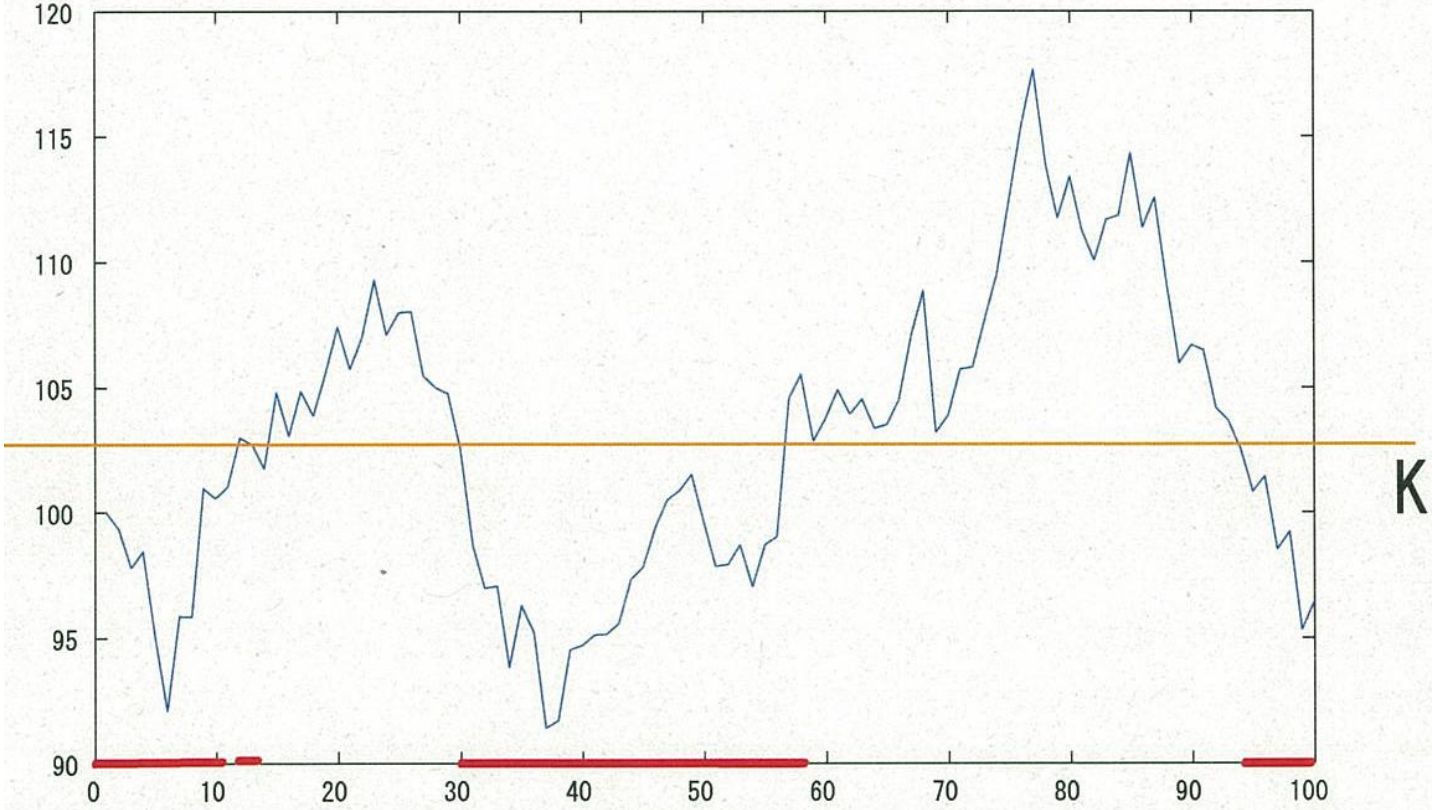
The relations of the Three " Nonparametric Statistics"

$$F(K)$$

$$F(S_t) = R(t)$$

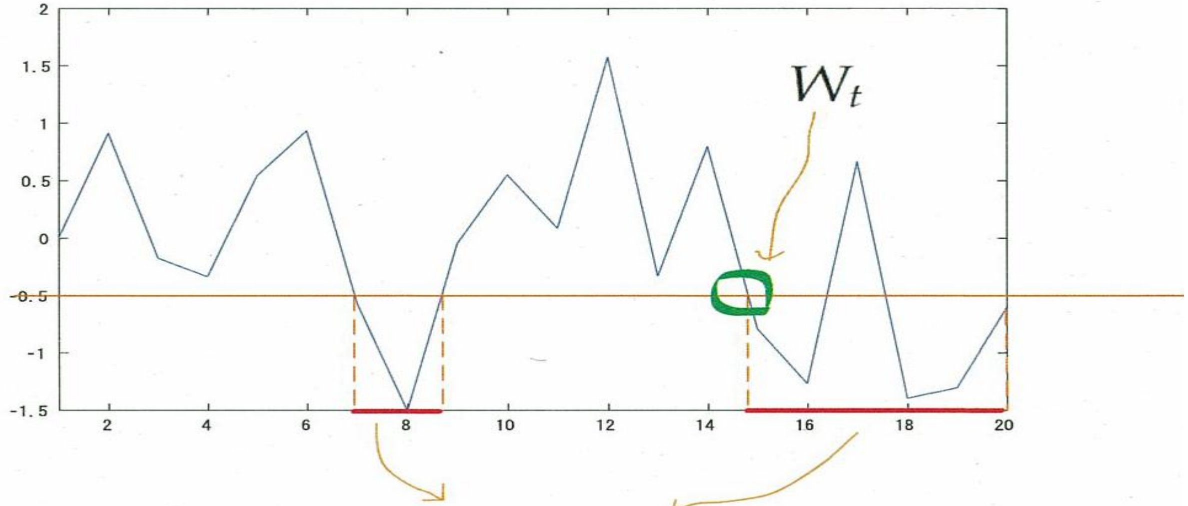
$$F(m(\alpha)) = \alpha$$

Fixed Corridor FC_K



$$FC_K := \int_0^T I\{S_u \leq K\} du$$

Rank of W_t



$$\int_0^T I\{W_u < W_t\}du$$

$$R^0_{t,T} := \frac{1}{T}\int_0^T I\{W_u < W_t\}dt$$

Derive Probability Distribution of Rank

S_t =exp($\mu t + \sigma W_t$)

Zero drift case: $\mu = 0$, $\sigma = 1$

Drift : $\mu \neq 0$, $\sigma > 0$

Derivation of the density function of $R^0_{t,T}$

$$
\begin{aligned}
R^0_{t,T} &= \frac{1}{T}\int_0^T I\{W_u < W_t\}du \\
&= \frac{1}{T}\int_0^t I\{W_t - W_u > 0\}du + \frac{1}{T}\int_t^T \{W_u - W_t < 0\}du \\
&= \frac{t}{T}\cdot\frac{1}{t}\int_0^t I\{\hat{W}_u > 0\}du + \frac{T-t}{T}\cdot\frac{1}{T-t}\int_0^{T-t} I\{\tilde{W}_u < 0\}du \\
&= \frac{t}{T}A_1 + \frac{T-t}{T}A_2
\end{aligned}
$$

A_1, A_2: two independent arcsine random variables

$\hat{W}_u := W_t - W_{t-u}$, $\tilde{W}_u := W_{t+u} - W_t$

$$E[h(T \bullet R^{*\mu,\sigma}_{t,T}(t))] = E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(T \bullet R^{*0,1}_{t,T}(t))]$$

$$= \iint_{-\infty<x<\infty, 0<y<1} e^{\frac{\mu}{\sigma}x - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(y) f_{(W_T, T \bullet R^{*0,1}_{t,T})}(x,y)dydx,$$

(more precisely, noting that $W_T = Z^*_{T-t} - Z_t$)

$$= E[e^{\frac{\mu}{\sigma}(Z^*_{T-t} - Z_t) - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(\int_0^t I\{Z_s \leq 0\}ds + \int_0^{T-t} I\{Z^*_s \leq 0\}ds)]$$

$$= \iint_{-\infty<x_1<\infty, 0<y_1<T-t} \iint_{-\infty<x_2<\infty, 0<y_2<t}$$

$$e^{\frac{\mu}{\sigma}(x_1 - x_2) - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(y_2 + y_1) f_{(Z^*_{T-t}, \int_0^{T-t} I\{Z^*_s \leq 0\}ds)}(x_1, y_1)dy_1dx_1 f_{(Z_t, \int_0^t I\{Z_s \leq 0\}ds)}(x_2, y_2)dy_2dx_2$$

Thus, it was enough to derive a joint probability distribution function (or density function) of ($W_{T,}$ $R^{*0,1}_{t,T}$) rather than that of (W_T, $R^{*\mu,\sigma}_{t,T}$) in order to calculate the above expectation. These joint densities of the decomposed variables;

$$f_{(Z^*_{T-t},\int_0^{T-t} I\{Z^*_s \le 0\}ds)}(x_1, y_1),\ f_{(Z_t,\int_0^{t} I\{Z_s \le 0\}ds)}(x_2, y_2)$$

can be obtained from Lemma1.

Lemma 1.

$$P(W_t \in da, \int_0^t I\{W_s < 0\}ds \in du)$$

$$= \begin{cases} \left(\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2(t-s)}} ds\right) da du & for\, a > 0 \\ \left(\int_0^u \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} e^{\frac{-a^2}{2s}} ds\right) da du & for\, a < 0 \end{cases}$$

Density function of $R^{\mu}_{t,T}$ (1)

$$f_{R^{\mu}_{t,T}}(x) = \int_{-\infty}^{\infty} \exp\left\{\mu y - \frac{\mu^2 T}{2}\right\} f_{(W_T, R_{t,T})}(y, x) dy$$

Density function of $R^{\mu}_{t,T}$ (2)

For $0 < t < T/2$,

$$f_{(W_T, R_{t,T})}(x, y) = T \iint_A f_t^+(x_1, y_1) f_{T-t}^-(x - x_1, Ty - y_1) dx_1 dy_1,$$

$A =$

$$\begin{cases} \{(x_1, y_1) : -\infty < x_1 < \infty, 0 < y_1 < Ty\}, \ -\infty < x < \infty, 0 < y < \frac{t}{T} \\ \{(x_1, y_1) : -\infty < x_1 < \infty, 0 < y_1 < t\}, \ -\infty < x < \infty, \frac{t}{T} < y < \frac{T-t}{T} \\ \{(x_1, y_1) : -\infty < x_1 < \infty, Ty - (T - t) < y_1 < t\}, \ -\infty < x < \infty, \frac{T-t}{T} < y < 1. \end{cases}$$

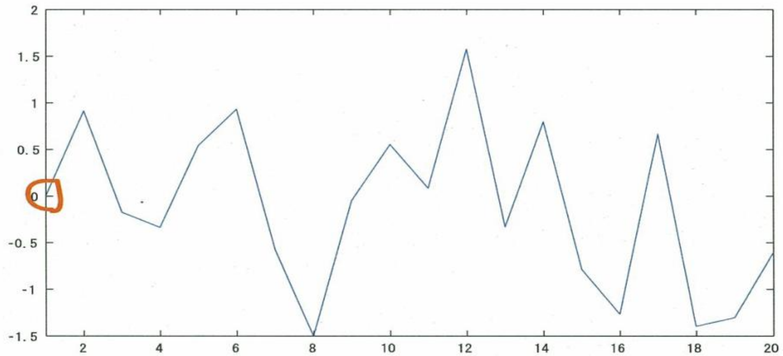
Density function of $R^{\mu}_{t,T}$ (3)

$$f_t^-(a,u) = \begin{cases} \displaystyle\int_u^t \frac{a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2(t-s)}\right\} ds, \ a > 0 \\ \displaystyle\int_0^u \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2s}\right\} ds, \ a < 0 \end{cases}$$

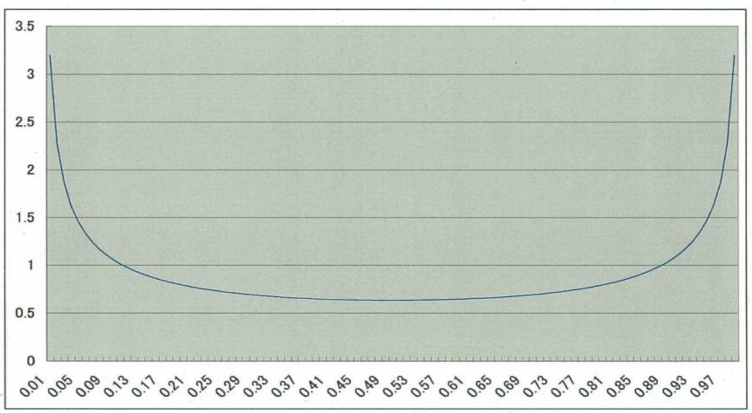
$$f_t^+(a,u) = \begin{cases} \displaystyle\int_0^u \frac{a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2s}\right\} ds, \ a > 0 \\ \displaystyle\int_u^t \frac{-a}{2\pi\sqrt{s^3(t-s)^3}} \exp\left\{-\frac{a^2}{2(t-s)}\right\} ds, \ a < 0 \end{cases}$$

f_t^-: joint density function of $(W_t, \int_0^t I\{W_u < 0\} du)$
f_t^+: joint density function of $(W_t, \int_0^t I\{W_u > 0\} du)$

$R^{\mu}_{0,1}$: rank of X_0

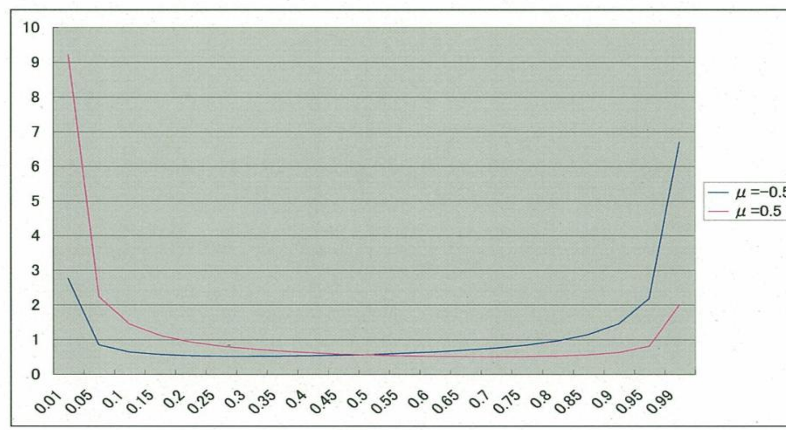


Density function of $R^{0}_{0,1}$



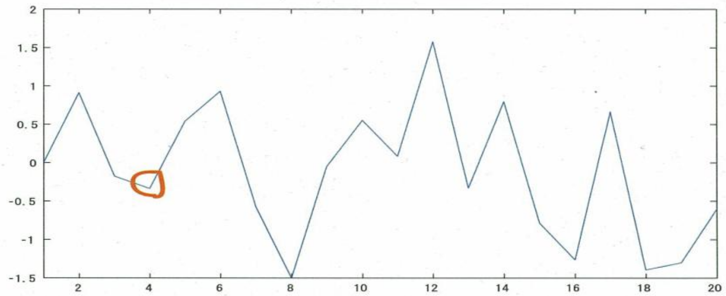
$R^{0}_{0,1} := \int_0^1 I\{W_u < W_0\}du$

Density function of $R^{0.5}_{0,1}$ and $R^{-0.5}_{0,1}$

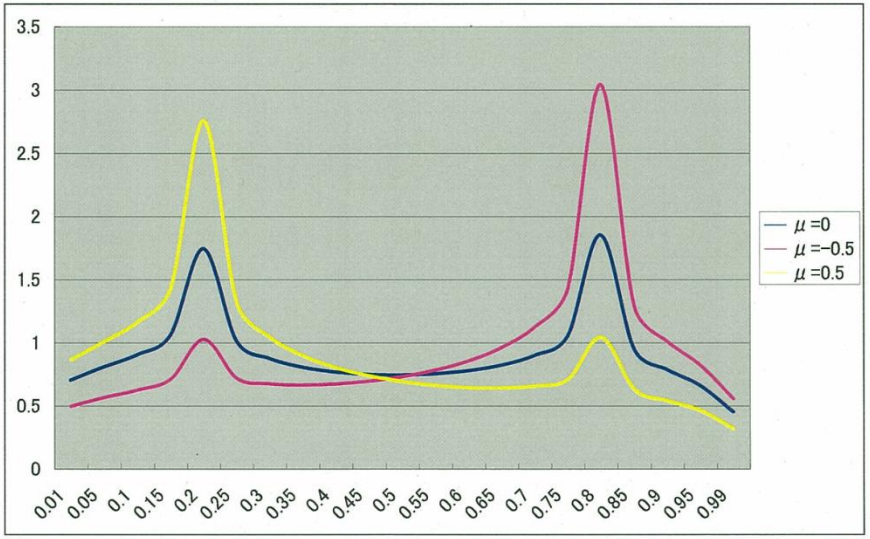


$R^{\mu}_{0,1} := \int_0^1 I\{X_u < X_0\}du,\ X_t = \mu t + W_t$

$R^{\mu}_{0.2,1}$: **rank of** $X_{0.2}$

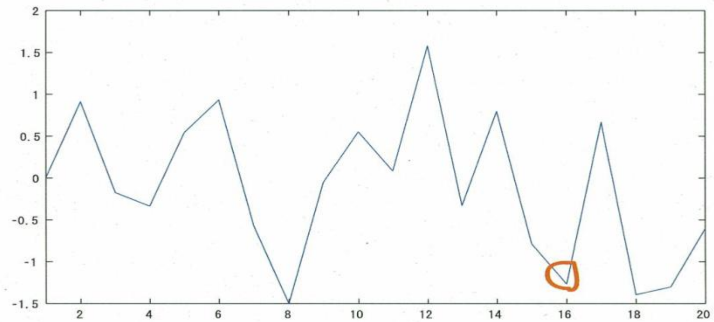


Density function of $R^{\mu}_{0.2,1}$

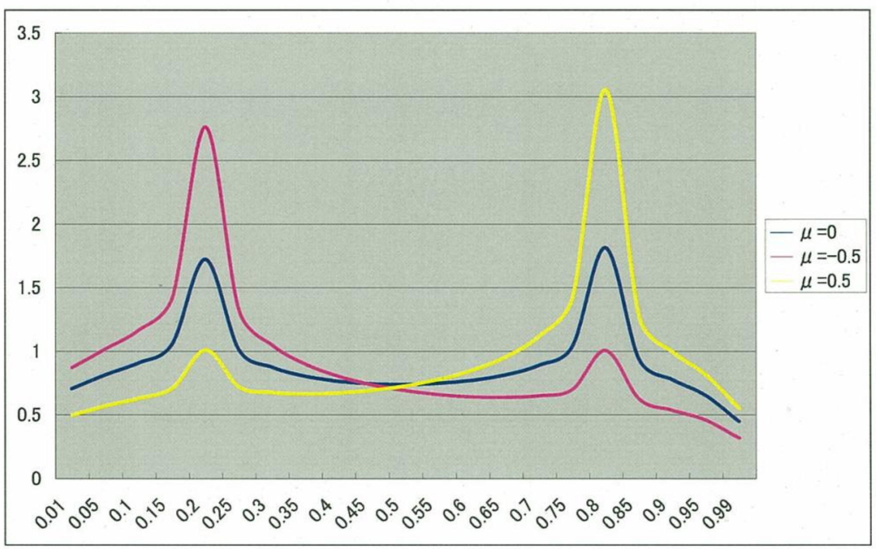


$$R^{\mu}_{0.2,1} := \int_0^1 I\{X_u < X_{0.2}\}du,\ X_t = \mu t + W_t$$

$R^{\mu}_{0.8,1}$: **rank of** $X_{0.8}$

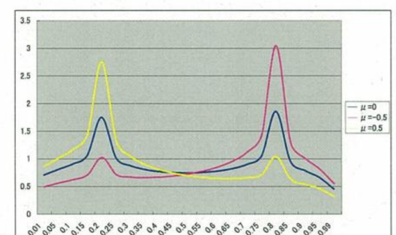


Density function of $R^{\mu}_{0.8,1}$

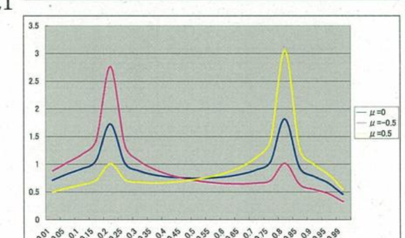


$$R^{\mu}_{0.8,1} := \int_0^1 I\{X_u < X_{0.8}\}du,\ X_t = \mu t + W_t$$

Comparison between $R^{\mu}_{0.2,1}$ and $R^{\mu}_{0.8,1}$

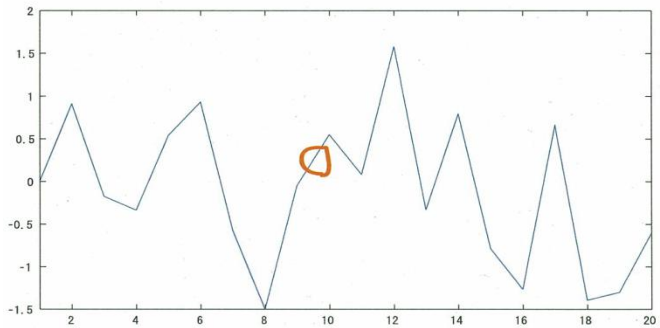


$R^{\mu}_{0.2,1}$

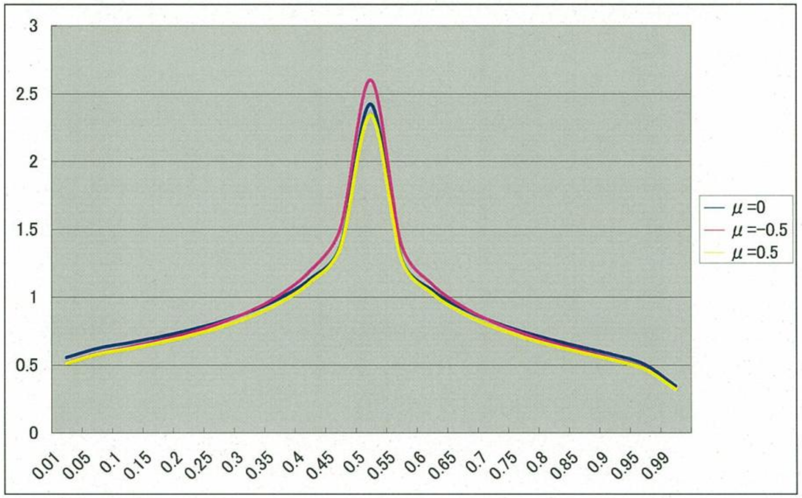


$R^{\mu}_{0.8,1}$

$R^{\mu}_{0.5,1}$: rank of $X_{0.5}$



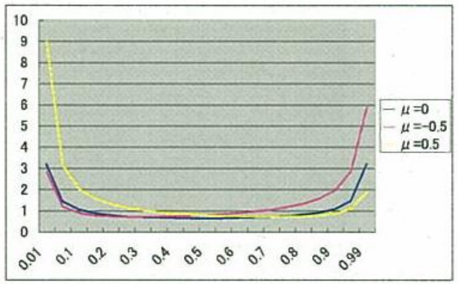
Density function of $R^{\mu}_{0.5,1}$



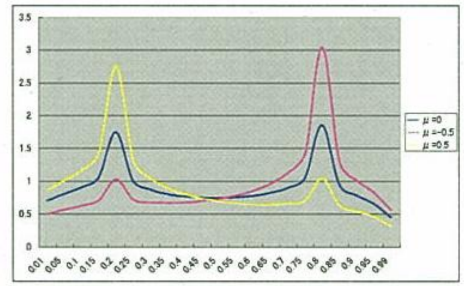
$$R^{\mu}_{0.5,1} := \int_0^1 I\{X_u < X_{0.5}\}du,\ X_t = \mu t + W_t$$

Summary

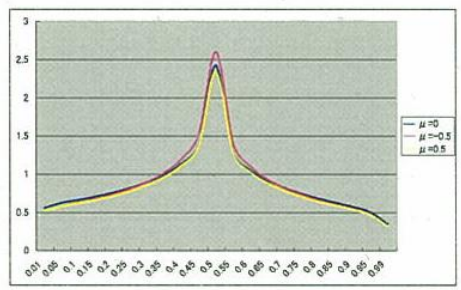
$t = 0$



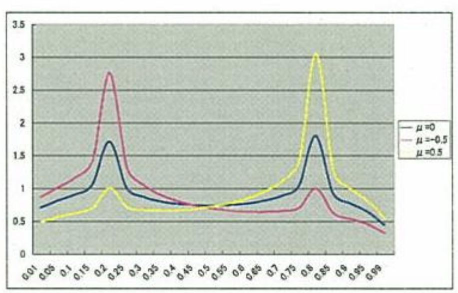
$t = 0.2$



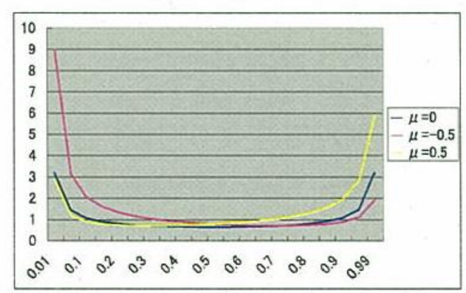
$t = 0.5$



$t = 0.8$



$t = 1$



Joint distribution of quantiles

: Fujita = two quantiles

: Imamura= more than two quantiles

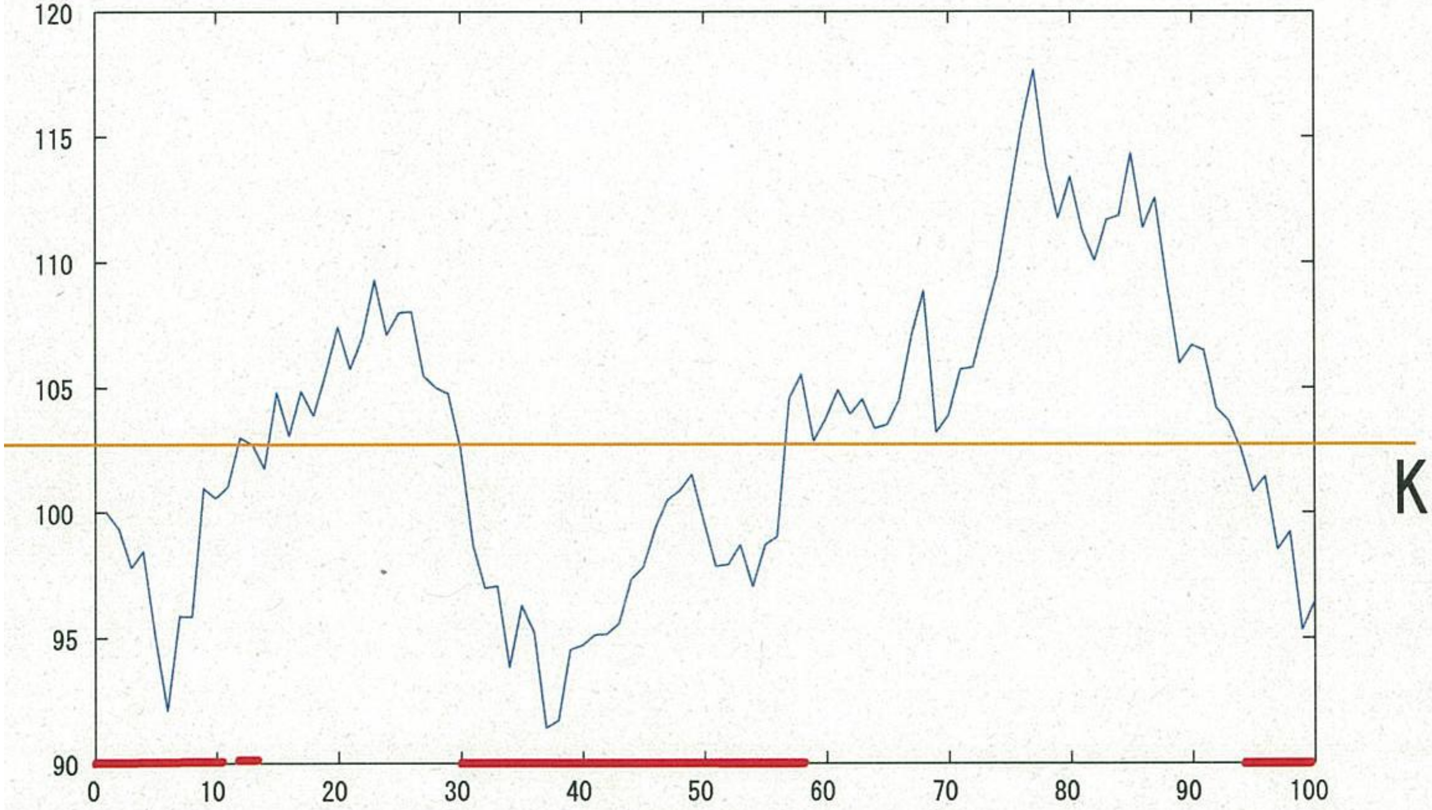
(decompose upper part and lower part and delete the flat part)

: Imamura+Miura= multivariate distribution in the form of each differences and maximum

: How about conditional distributions given Maximum and /or Minimum

Section 2. F(K) and Rank

Fixed Corridor FC_K



$$FC_K := \int_0^T I\{S_u \leq K\} du$$

$$P\{TF(K) < x\}$$

$$= P\{\int_0^T I\{S_u \le K\} du < x\}$$

$$= P\{\int_0^T I\{\frac{\mu}{\sigma} u + W_u \le A\} du < x\} = E[I\{\int_0^T I\{\frac{\mu}{\sigma} u + W_u \le A\} du < x\}]$$

$$= E[e^{\frac{\mu}{\sigma} W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} I\{\int_0^T I\{W_u \le A\} du < x\}] = E[e^{\frac{\mu}{\sigma} W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} I\{(\tau + \int_\tau^T I\{Z_{u-\tau} \le 0\} du) < x\}]$$

(where we put $Z_{u-\tau} = W_u - W_\tau$ and $\tau = \inf\{u: W_u \ge A\}$)

$$= E[e^{\frac{\mu}{\sigma} W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} e^{-Z_{T-\tau} + Z_{T-\tau}} I\{(\tau + \int_0^{T-\tau} I\{Z_u \le 0\} du) < x\}]$$

$$= E[e^{\frac{\mu}{\sigma} A - (\frac{\mu}{\sigma})^2 \frac{T}{2}} e^{Z_{T-\tau}} I\{(\tau + \int_0^{T-\tau} I\{Z_u \le 0\} du) < x\}]$$

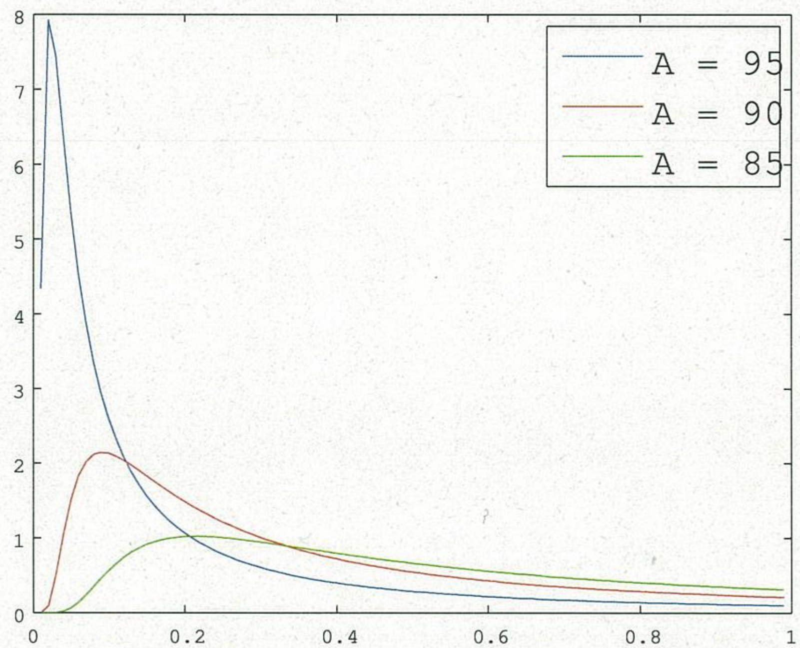
which can be calculated by using the joint probability density function of

$$(Z_{T-\tau}, \int_0^{T-\tau} I\{Z_u \le 0\} du < x)$$

shown in Lemma1, and the distribution of the first hitting time τ of W_u to A

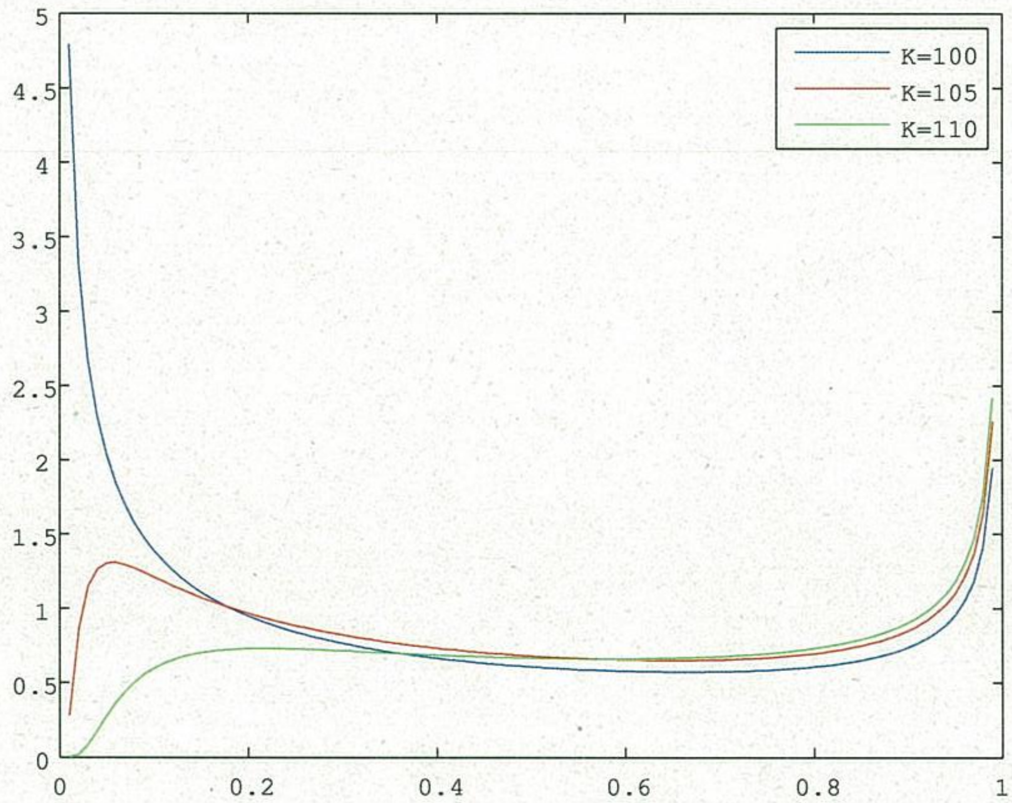
- $\tau_A := \inf\{t > 0 \ : \ S_t = A\}$

Density function of τ_A



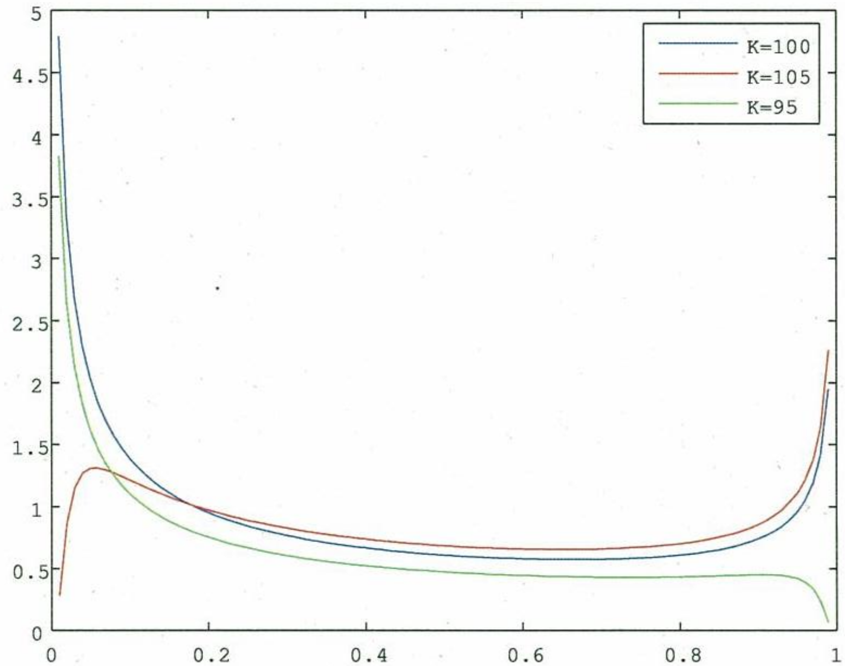
$\tau_A := \inf\{t > 0 : \ S_t = A\}$, $S_t = S_0 \exp\left\{(r - \frac{\sigma^2}{2})t + \sigma W_t\right\}$ $(S_0 = 100)$

Density function of $\int_0^1 I\{S_u < K\}du$



$$S_t = S_0 \exp\left\{(r - \frac{\sigma^2}{2})t + \sigma W_t\right\} \ (S_0 = 100)$$

Density Function of Fixed Corridor



$$FC_K := \int_0^T I\{S_u \le K\}du,\ S_t = S_0 \exp\left\{(r - \frac{\sigma^2}{2})t + \sigma W_t\right\}$$

$$S_0 = 100, r = 0.1, \sigma = 0.2, T = 1$$

: (2-3). Probability Distribution of α -quantile of Brownian Moton

α -quantile is defined as a quantity m(α) for any given $\alpha \in [0, 1]$ such that

$\alpha = \frac{1}{T}\int_0^T I\{S_u \le m(\alpha)\}du$

i.e. α-quantile is the level below which
the stock price stays for 100α-percent of time during the time interval [0,T].

Based on the relation; for any y≧0,

$$\{\frac{1}{T}\int_0^T I\{S_u \le y\}du < \alpha\} = \{m(\alpha) > y\}$$

the probability distribution for m(α) can be obtained as

$$P\{m(\alpha) < y\} = 1 - P\{\frac{1}{T}\int_0^T I\{S_u \le y\}du < \alpha\}$$

Akahori(1995) rigorously derived the probability distribution of m(α),
and Fujita(1997) derived the joint probability density function of

$$(S_T, m(\alpha))$$

in order to price a call option with payoff function; max{S_T - m(α),0}.

Alternative approach to derive the density

Dassios(1995)

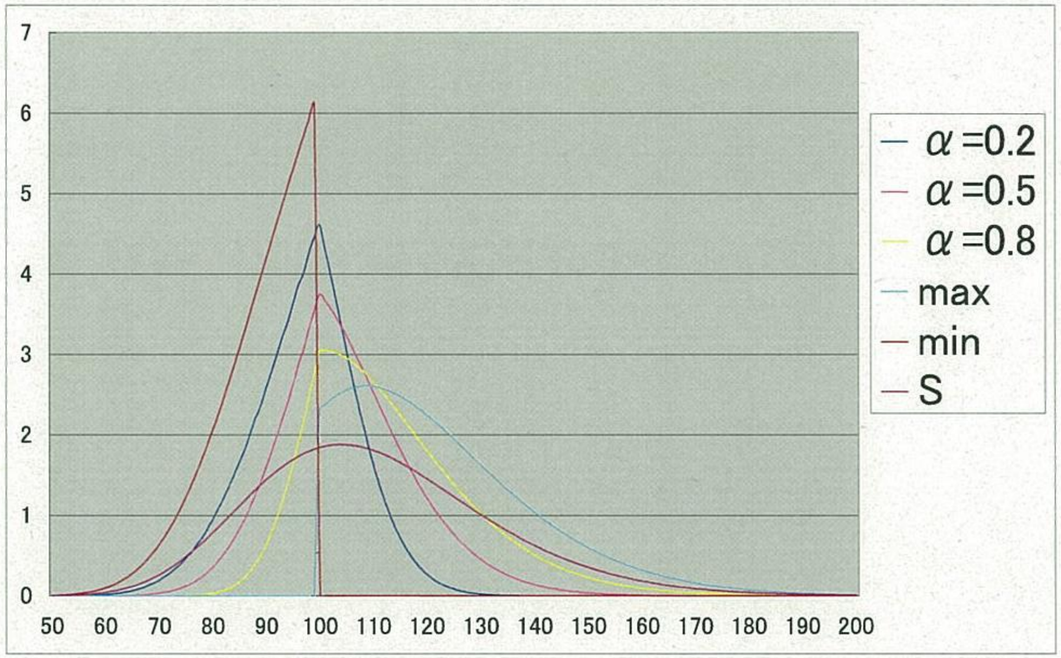
$$m(\alpha : [0,t]) = \text{(in Law)} = Y_1 + Y_2 ,$$

where Y_1 and Y_2 are independent, and

$$Y_1 = \text{(in Law)} = \sup_{0<u<\alpha t} X_u$$

$$Y_2 = \text{(in Law)} = \sup_{0<u<(1-\alpha)t} X_u$$

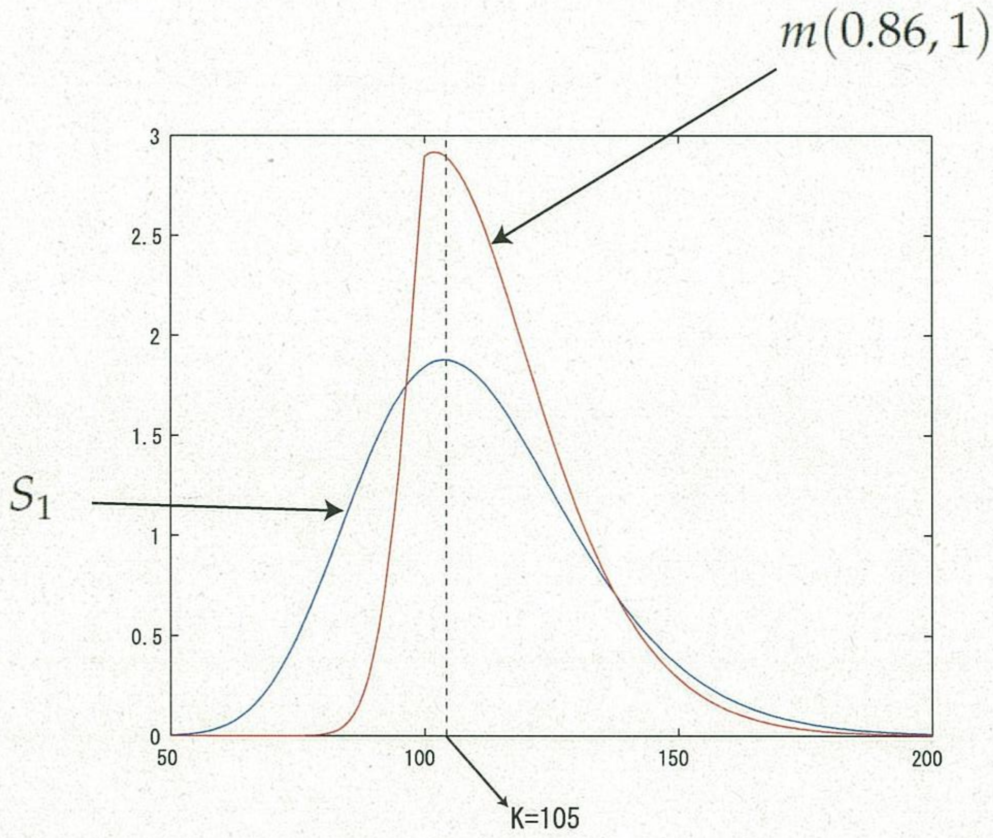
Density function of α-quantile



$$m(\alpha, T) := \inf\left\{x \ : \ \frac{1}{T}\int_0^T I\{S_u \leq x\}du > \alpha\right\}$$

$$S_t = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}, S_0 = 100, r = 0.1, \sigma = 0.2, T = 1$$

Density function of S_1 and $m(0.86, 1)$



- $S_t = S_0 \exp\left\{(r - \frac{\sigma^2}{2})t + \sigma W_t\right\}, r = 0.1, \sigma = 0.2, T = 1, S_0 = 100, t \in [0, T]$

Items & Tools

First Hitting Time

Arcsine Law

Joint density (Fujita):occupation time

:************************

Feynman=Kac Theorem

Laplace Transform

Cameron=Martin=Girsanov=Maruyama

1. First hitting time

Refer to

Wellner &Shorack

Or

Karatzas & Shreve.

And give a brief description in the class

We now record various probabilities associated with Brownian motion:

$$P\left(\sup_{0\le s\le t} \mathbb{S}(s) > b\right) = 2P(N(0, t) > b) \tag{6}$$

Consider (6). Corresponding to every sample path having $\mathbb{S}(t) > b$, there are exactly two "equally likely" sample paths (see Figure 2) for which $\|\mathbb{S}^+\|_0^t > b$. Since $\mathbb{S}(t) \cong N(0, t)$, Equation (6) follows. The theorem that validates our key step is the "strong Markov property" (see Theorem 2.5.1); we paraphrase it by saying that if one randomly stops Brownian motion at a random time τ that depends only on what the path has done so far, then what happens after time τ as measured by $\{\mathbb{S}(\tau+t) - \mathbb{S}(\tau), t \ge 0\}$ has exactly the same distribution as does the Brownian motion $\{\mathbb{S}(t): t \ge 0\}$. In the argument above, τ was the first time that $\mathbb{S}$ touches the line $y = b$. Change variables to get the second formula.

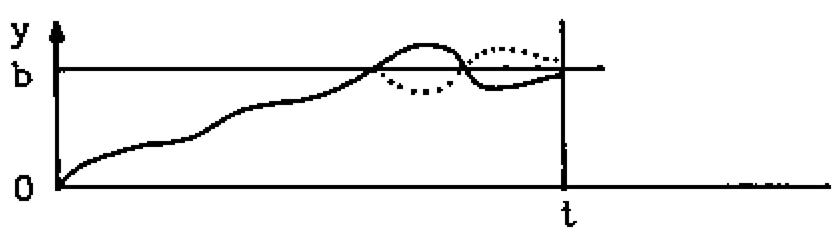


Figure 2.

We now record various probabilities associated with Brownian motion: $\mathbb{S}$

$$P\left(\sup_{0\le s\le t} \mathbb{S}(s) > b\right) = 2P(N(0,t) > b) \tag{6}$$

$$= \int_0^t \frac{b}{\sqrt{2\pi s^3}} \exp\left(-\frac{b^2}{2s}\right) ds \qquad \text{for all } b > 0$$

$$= F_\tau(t)$$

where $\tau \equiv \inf\{s: \mathbb{S}(s) = b\}$;

$\{\tau \le t\} \Leftrightarrow \{\sup_{0\le s\le t} \mathbb{S}(s) > b\}$

$$2P(N(0,t) > b) = 2\int_b^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \int_0^t \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} ds$$

by change of variable: s= $\frac{tb^2}{x^2}$, then, $-\frac{b}{2s^{\frac{3}{2}}} ds = \frac{1}{\sqrt{t}} dx.$

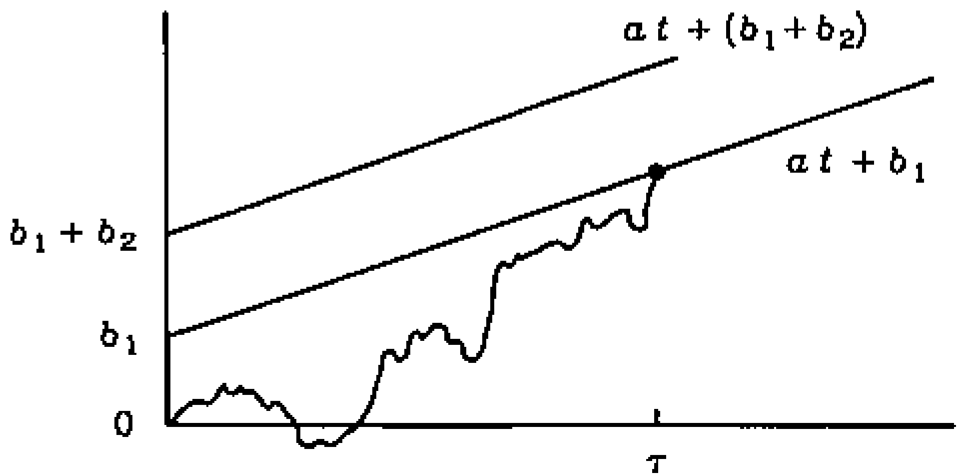


Figure 4.

Consider (8). We follow Doob (1949). Let

(d) $$\phi(a, b) \equiv P(\mathbb{S}(t) \geq at + b \text{ for some } t \geq 0) \qquad \text{where } a \geq 0,\ b > 0.$$

Now note that

(e) $$\phi(a, b_1 + b_2) = \phi(a, b_1)\phi(a, b_2)$$

because at the instant τ when $\mathbb{S}$ first hits the line $at + b_1$, we then have the same problem all over again where the equation of the original line $at + (b_1 + b_2)$ relative to the point $(\tau, \mathbb{S}(\tau))$ has become $at + b_2$ (see Figure 4). This is again rigorized by the strong Markov property. Also, $\phi(a, b) \geq P(\mathbb{S}(1) \geq a + b) > 0$ and $\phi(a, b)$ is $\searrow$ in b. The only solution in b of the functional equation (e) having these properties is

(f) $$\phi(a, b) = \exp(-\psi(a)b) \qquad \text{for some constant } \psi(a) \text{ depending on } a.$$

Note from Figure 5 that once $\mathbb{S}$ intersects $y=b$ at time τ, the event that is then required has probability $\phi(a, a\tau)$ by the strong Markov property. We thus have

$$\exp(-\psi(a)b) = \phi(a, b) \qquad \text{by (f)}$$

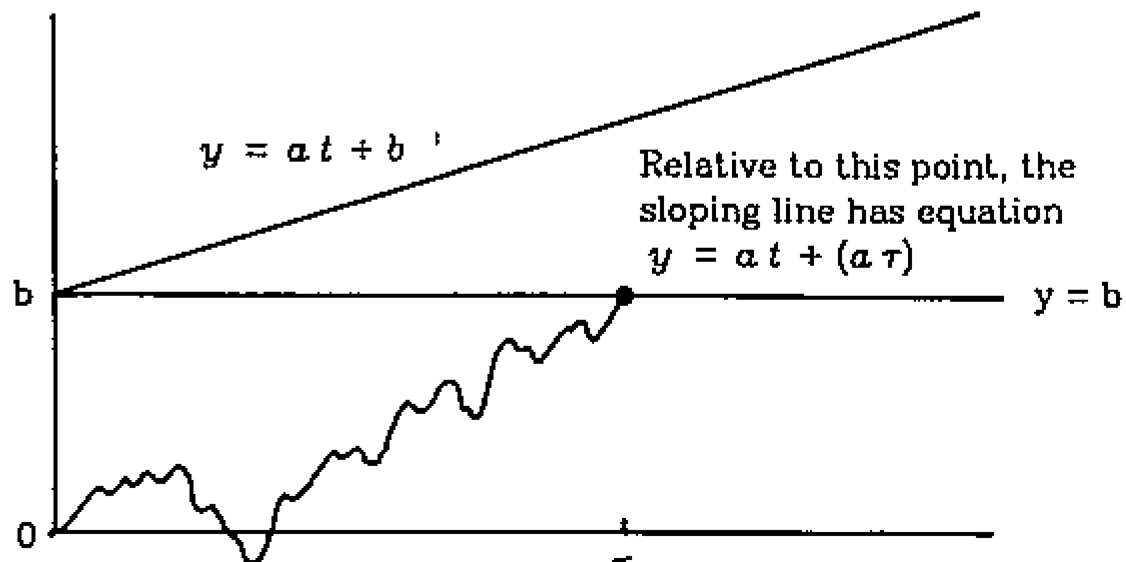
$$\text{(g)} \qquad = E_\tau[\phi(a, a\tau)] \qquad \text{by the strong Markov property}$$

$$\text{(h)} \qquad = \int_0^\infty \exp(-\psi(a)as)\frac{b}{\sqrt{2\pi}s^{3/2}}\exp\left(-\frac{b^2}{2s}\right) ds \qquad \text{by (f) and (14)}$$

$$= \frac{2}{\pi}\int_0^\infty \exp\left(-\frac{a\psi(a)b^2}{2y^2}+y^2\right) dy \text{ letting } y^2 = \frac{b^2}{2s}$$

$$= \exp(-b\sqrt{2a\psi(a)}) \qquad \text{by elementary integration,}$$

so that $\psi(a) = 2a$.



y = a t + b
Relative to this point, the
sloping line has equation
y = a t + (a τ)
b
y = b
0
τ

Figure 5.

2. Cameron-Martin-Girsanov-Maruyama Theorem

Brief Description of the theorem.

Please take it from a book of Probability for a rigorous and general form.

C-M-G-M(Maruyama) Theorem.(a brief form)

(Ω, F, P)

Let,

W_t , t $\in [0,\infty)$: Brownian Motion under P.

γ_t , t $\in [0,\infty)$: $\mathrm{F}_t - \textit{previsible}$ process with $\mathrm{E}^P[e^{\frac{1}{2}\int_0^T \gamma_t^2 dt}] < \infty.$

Then, $\exists$ Q such that

(i) Q and P are equivalent.

(ii) $\dfrac{dQ}{dP} = e^{\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt}$

(iii) $\widetilde{W_t} = \mathrm{W}_t + \int_0^t \gamma_u du$ is a Brownian Motion under Q.

:*****************************

In application,

$$\mathrm{E}^P[h(\mathrm{W}_t + \int_0^t \gamma_u du)] = \mathrm{E}^P[e^{\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt} h(W_t)]$$

$$E[h(T \bullet R_{t,T}^{*\mu,\sigma}(t))] = E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(T \bullet R_{t,T}^{*0,1}(t))]$$

$$E[h(T \bullet R_{t,T}^{*\mu,\sigma}(t))] = E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(T \bullet R_{t,T}^{*0,1}(t))]$$

$= \iint_{-\infty<x<\infty, 0<y<1} e^{\frac{\mu}{\sigma}x - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(y) f_{(W_T, T \bullet R_{t,T}^{*0,1})}(x,y)dydx$, (more precisely, noting that $W_T = Z_{T-t}^* - Z_t$)

$$= E[e^{\frac{\mu}{\sigma}(Z_{T-t}^* - Z_t) - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(\int_0^t I\{Z_s \le 0\}ds + \int_0^{T-t} I\{Z_s^* \le 0\}ds)]$$

$$= \iint_{-\infty<x_1<\infty, 0<y_1<T-t} \iint_{-\infty<x_2<\infty, 0<y_2<t}$$

$$e^{\frac{\mu}{\sigma}(x_1 - x_2) - (\frac{\mu}{\sigma})^2 \frac{T}{2}} h(y_2 + y_1) f_{(Z_{T-t}^*, \int_0^{T-t} I\{Z_s^* \le 0\}ds)}(x_1, y_1)dy_1dx_1 f_{(Z_t, \int_0^t I\{Z_s \le 0\}ds)}(x_2, y_2)dy_2dx_2$$

$$P\{TF(K) < x\}$$

$$= P\{\int_0^T I\{S_u \le K\}du < x\}$$

$$= P\{\int_0^T I\{\frac{\mu}{\sigma}u + W_u \le A\}du < x\} = E[I\{\int_0^T I\{\frac{\mu}{\sigma}u + W_u \le A\}du < x\}]$$

$$= E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} I\{\int_0^T I\{W_u \le A\}du < x\}] = E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} I\{(\tau + \int_\tau^T I\{Z_{u-\tau} \le 0\}du) < x\}]$$

(where we put $Z_{u-\tau} = W_u - W_\tau$)

$$= E[e^{\frac{\mu}{\sigma}W_T - (\frac{\mu}{\sigma})^2 \frac{T}{2}} e^{-Z_{T-\tau} + Z_{T-\tau}} I\{(\tau + \int_0^{T-\tau} I\{Z_u \le 0\}du) < x\}]$$

$$= E[e^{\frac{\mu}{\sigma}A - (\frac{\mu}{\sigma})^2 \frac{T}{2}} e^{Z_{T-\tau}} I\{(\tau + \int_0^{T-\tau} I\{Z_u \le 0\}du) < x\}]$$

which can be calculated by using the joint probability density function of

$$(Z_{T-\tau}, \int_0^{T-\tau} I\{Z_u \le 0\}du < x)$$

shown in Lemma1, and the distribution of the first hitting time τ of W_u to A

3. Levy’s Arc Sine Law

Use Feynman-Kac Theorem and Laplace Transformation, to derive it.

Show some details of derivation (calculation).

Karatzas & Shreve has a description.

Feynman-Kac Formula.(brief memo)

The parabolic equation for u(t,x) (t,x) $\in$[0,∞)x(-∞,∞).

$u_t + k(.)u = \frac{1}{2}u_{xx}$ with u(0,x)=f(x).

Assume: $\lim_{t\to\infty} e^{-\alpha t}u(t,x) = 0$ for α>0 and x $\in$(-∞,∞).

Laplace transformed function $z_\alpha(x) = \int_0^\infty e^{-\alpha t}u(t,x)dt,\ \alpha>0$

satisfies $\frac{1}{2}\Delta z_\alpha(.) = (\alpha + k(.))z_\alpha(.) - f(.).$

:*******************************

Theorem 4.9.

Let,

$f : R \to R.$ with $\int_{-\infty}^{\infty} |f(x+y)| e^{-|y|\sqrt{2\alpha}} dy < \infty.$ (or $E^x[\int_0^\infty e^{-\alpha t} |f(W_t)| dt] < \infty.$)$\forall x$

$k : R \to (0,\infty)$; *piecewise continuous*

Then,

$z(x) = E^x[\int_0^\infty e^{-\alpha t} f(W_t) e^{-\int_0^t k(W_s)ds} dt]$ is piecewise C^2 and satisfies

$\frac{1}{2}z''(.) = (\alpha + k(.))z(.) - f(.)$ on $R \backslash (D_f \cup D_k)$

Karatzas&Shreve

Proposition 4.11.(Levy's Arc-Sine Law for the occupation time of (0,∞).)

W_s ;Wiener Process.

$0<\theta<t$,

$$P^0\left[\int_0^t 1_{(0,\infty)}(W_s)ds < \theta \right] = \int_0^{\frac{\theta}{t}} \frac{1}{\pi\sqrt{s(1-s)}}ds = \frac{2}{\pi}\arcsin\sqrt{\frac{\theta}{t}}$$

Proof.

Put $k(.)=\beta 1_{(0,\infty)}(.)$ and $f(.)=1$ in Theorem 4.9. The theorem says,

$$z(x) = E^x\left[\int_0^\infty e^{-\alpha t} e^{-\beta\int_0^t 1_{(0,\infty)}(W_s)ds} dt\right],\ (W_0 = x.) \text{ satisfies}$$

$$\frac{1}{2}z''(.) = (\alpha + \beta 1_{(0,\infty)}(.))z(.) - 1.$$

Then, solve this, and calculate to find

$$\int_0^\infty e^{-\alpha t}\int_0^t e^{-\beta y} g(y)dy)dt = \frac{1}{\sqrt{\alpha(\alpha+\beta)}},$$

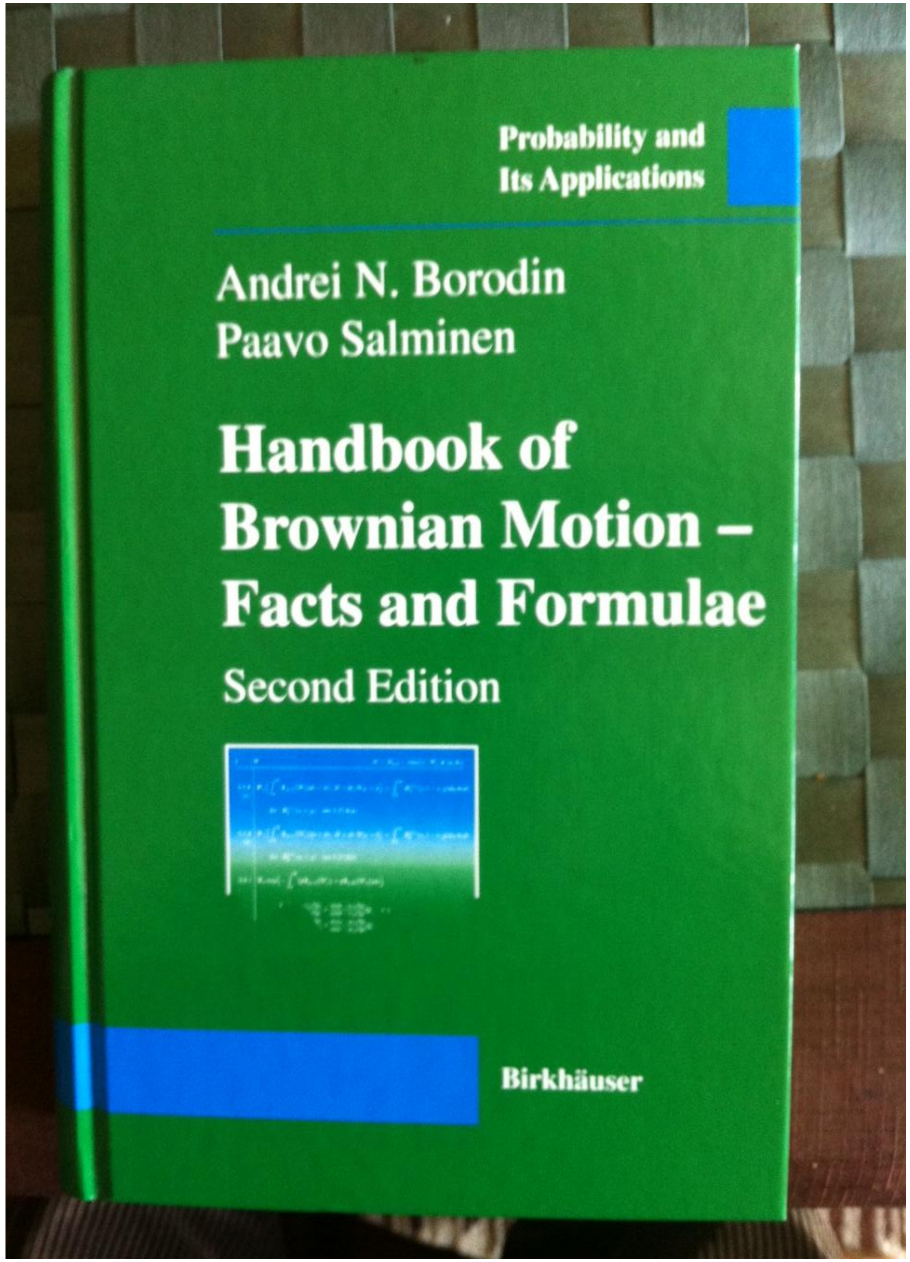
where g(.) is the density function of $\int_0^t 1_{(0,\infty)}(W_s)ds$.

Some calculation gives $g(y)=\dfrac{1}{\pi\sqrt{y(t-y)}}$.

4. Joint Probability Density. Lemma 1.

an outline of derivation is shown, by Fujita in his working paper and Fujita&Miura's paper (Edokko Options(2003)).

Also, we see it in other literatures as well.


Probability and
Its Applications
Andrei N. Borodin
Paavo Salminen
Handbook of
Brownian Motion –
Facts and Formulae
Second Edition
Birkhäuser

(2) $r \le x$	$\mathbf{P}_x\Big(\int_0^t \mathbb{I}_{[r,\infty)}(W_s)ds = t\Big) = 1 - \operatorname{Erfc}\Big(\frac{x-r}{\sqrt{2t}}\Big)$
1.4.5	$\mathbf{E}_x\Big\{\exp\Big(-\gamma\int_0^\tau \mathbb{I}_{[r,\infty)}(W_s)ds\Big); W_\tau \in dz\Big\} =: \lambda Q_\lambda^{(4)}(z)dz$
$x \le r$ $z \le r$	$= \frac{\lambda}{\sqrt{2\lambda}}e^{-\lvert z-x\rvert\sqrt{2\lambda}}dz + \lambda\Big(\frac{2}{\sqrt{2\lambda}+\sqrt{2\lambda+2\gamma}} - \frac{1}{\sqrt{2\lambda}}\Big)e^{(z+x-2r)\sqrt{2\lambda}}dz$
$x \le r$ $r \le z$	$= \frac{2\lambda}{\sqrt{2\lambda}+\sqrt{2\lambda+2\gamma}}e^{(x-r)\sqrt{2\lambda}+(r-z)\sqrt{2\lambda+2\gamma}}dz$
$r \le x$ $z \le r$	$= \frac{2\lambda}{\sqrt{2\lambda}+\sqrt{2\lambda+2\gamma}}e^{(r-x)\sqrt{2\lambda+2\gamma}+(z-r)\sqrt{2\lambda}}dz$
$r \le x$ $r \le z$	$= \frac{\lambda}{\sqrt{2\lambda+2\gamma}}\Big(e^{-\lvert z-x\rvert\sqrt{2\lambda+2\gamma}} + \Big(1 - \frac{2\sqrt{2\lambda}}{\sqrt{2\lambda+2\gamma}+\sqrt{2\lambda}}\Big)e^{(2r-x-z)\sqrt{2\lambda+2\gamma}}\Big)dz$
1.4.6	$\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W_s)ds \in dy, W_\tau \in dz\Big) =: \lambda B_\lambda^{(4)}(y,z)dydz$
$x \le r$ $z \le r$	$= \lambda\Big(\frac{\sqrt{2}}{\sqrt{\pi y}}e^{-\lambda y} - \sqrt{2\lambda}\operatorname{Erfc}(\sqrt{\lambda y})\Big)e^{(z+x-2r)\sqrt{2\lambda}}dydz$
$x \le r$ $r \le z$	$= \Big(\frac{\lambda\sqrt{2}}{\sqrt{\pi y}}e^{(x-r)\sqrt{2\lambda}-\lambda y-(z-r)^2/2y} - \sqrt{2\lambda}^3 e^{(z+x-2r)\sqrt{2\lambda}}\operatorname{Erfc}\Big(\frac{z-r}{\sqrt{2y}}+\sqrt{\lambda y}\Big)\Big)dydz$
$r \le x$ $z \le r$	$= \Big(\frac{\lambda\sqrt{2}}{\sqrt{\pi y}}e^{(z-r)\sqrt{2\lambda}-\lambda y-(x-r)^2/2y} - \sqrt{2\lambda}^3 e^{(z+x-2r)\sqrt{2\lambda}}\operatorname{Erfc}\Big(\frac{x-r}{\sqrt{2y}}+\sqrt{\lambda y}\Big)\Big)dydz$
$r \le x$ $r \le z$	$= \lambda\Big(\frac{1}{\sqrt{2\pi y}}e^{-\lambda y}\big(e^{-(z-x)^2/2y} + e^{-(2r-z-x)^2/2y}\big)$
	$-\sqrt{2\lambda}e^{(z+x-2r)\sqrt{2\lambda}}\operatorname{Erfc}\Big(\frac{z+x-2r}{\sqrt{2y}} + \sqrt{\lambda y}\Big)\Big)dydz$
(1) $x \le r$ $z \le r$	$\mathbf{P}_x\Big(\int_0^\tau \mathbb{I}_{[r,\infty)}(W_s)ds = 0,\ W_\tau \in dz\Big) = \frac{\sqrt{\lambda}}{\sqrt{2}}\big(e^{-\lvert z-x\rvert\sqrt{2\lambda}} - e^{(z+x-2r)\sqrt{2\lambda}}\big)dz$

Fujita also derived it in Fujita&Miura(2003)

The approach is the same as that for the occupation time : Arc Sin Law; using Feynman-Kac Theorem.

Homework for students.

Please submit next Monday to the office of ?

: (1) $0<T_1<<T<\infty$. $S_{t\ ,}$, t on [0,T] a Geometric Brownian motion.

Are the followings true? If true, prove it. If not true, give a counter example.

: (1-1).For each path, if $m(\alpha: [0, T]) < m(\alpha: [0, T_1])$, then $m(\alpha: [0, T]) > m(\alpha: [T_1, T])$.

: (1-2). $\{Rank(S_{T1}: [0, T_1]) > \alpha\} = \{S_{T1} > m(\alpha: [0, T_1])\}$

: (1-3). For each path, Let $\alpha(\omega) = Rank(S_{T1}: [0, T_1])(\omega)$.

If $Rank(S_{T1}: [0, T_1]) > Rank(S_{T1}: [0, T])$, then $m(\alpha(\omega): [0, T]) > S_{T1}$.

: (2). Use Brownian quantiles $m(\alpha:[0,T])$ for $0<\alpha<1$ to express the area under a path of a geometric Brownian motion, roughly.