Řešení úloh 4. soutěžní série

1. From the given vectors $v_1, \dots, v_{n+2}$, form vectors $w_1, \dots, w_{n+2} \in \mathbb{R}^{n+1}$ so that their first n coordinates are the same as those of corresponding v_i's and their last coordinate is 1. The vectors w_i being linearly dependent (there is more of them then the dimension of the space), there is a non-trivial linear combination resulting in zero. This equality in the last coordinate forces the sum of coefficients to be zero as desired.

2. Firstly,

$$k\binom{s}{k} = s\binom{s-1}{k-1},$$

so we need to compute

$$\sum_k \binom{n}{k}\binom{s-1}{k-1}.$$

However, this is just a special case of the problem 3.3 (domácí série), thus the result is

$$s\binom{n+s-1}{n-1}.$$

3. **Second solution.** Let $\mathcal{P}$ be the set of prime numbers, and for an infinite subset $\Sigma = \{p_0, p_1, \dots\}$ of $\mathcal{P}$ arranged in increasing order assign

$$A_\Sigma = \{p_0, p_0p_1, p_0p_1p_2, \dots\}.$$

The prime factorization for integers is unique, hence if $\Sigma' \subseteq \mathcal{P}$ is another infinite subset of $\mathcal{P}$ different from Σ, then A_Σ and $A_{\Sigma'}$ have only finitely many common terms. Since the number of different Σ's is continuum (see Problems 5 and 2.4), we are done.

Third solution. It is sufficient to show the result for $\mathbf{Q}$ rather than for $\mathbf{N}$, i.e., that there are continuum many sets $A_\gamma \subset \mathbf{Q}$ such that if $\gamma_1 \neq \gamma_2$, then $A_{\gamma_1} \cap A_{\gamma_2}$ is a finite set. Now choose for every $\gamma \in \mathbf{R}$ a rational sequence $A_\gamma = \{r_k^{(\gamma)}\}_{k=0}^{\infty}$ converging to γ. These A_γ sets clearly satisfy the requirements, for two sequences converging to different limits can have only finitely many terms in common.

4.

$$(2) \quad \left(1+\frac{1}{M}\right)\left(1+\frac{1}{M^2}\right)\left(1+\frac{1}{M^4}\right)\cdots = 1+\frac{1}{M}+\frac{1}{M^2}+\frac{1}{M^3}+\cdots = 1+\frac{1}{M-1}.$$

b) From (2) α is rational if its product ends in the stated way.

Conversely, suppose α is the rational number $\frac{p}{q}$. Our aim is to show that for some m,

$$\theta_{m-1} = \frac{n_m}{n_m - 1}. \qquad \theta_k = \frac{\alpha}{\prod_1^k (1+\frac{1}{n_i})}$$

Suppose this is not the case, so that for every m,

$$(3) \qquad \theta_{m-1} < \frac{n_m}{n_m - 1}.$$

For each k we write

$$\theta_k = \frac{p_k}{q_k}$$

as a fraction (not necessarily in lowest terms) where

$$p_0 = p, \quad q_0 = q$$

and in general

$$p_k = p_{k-1}n_k, \quad q_k = q_{k-1}(n_k+1).$$

The numbers $p_k - q_k$ are positive integers: to obtain a contradiction it suffices to show that this sequence is strictly decreasing. Now,

$$\begin{aligned} p_k - q_k - (p_{k-1} - q_{k-1}) &= n_k p_{k-1} - (n_k+1)q_{k-1} - p_{k-1} + q_{k-1} \\ &= (n_k - 1)p_{k-1} - n_k q_{k-1} \end{aligned}$$

and this is negative because

$$\frac{p_{k-1}}{q_{k-1}} = \theta_{k-1} < \frac{n_k}{n_k - 1}$$

by inequality (3).