Řešení úloh 2. soutěžní série

1. Dokážeme, že v požadovaném tvaru jdou napsat všechna čísla, která nejsou tvaru 2^m pro nějaké celé číslo $m \geq 0$.

To, že je nějaké číslo n napsané jako součet několika po sobě jdoucích čísel, znamená, že $n = (k+1)+(k+2)+\cdots+l$ pro nějaká čísla $0 \leq k < l-1$. Toto vyjádření můžeme pomocí známého vzorce upravit do tvaru $n = (1+2+\cdots+l)-(1+2+\cdots+k) = \frac{l(l+1)}{2} - \frac{k(k+1)}{2} = \frac{(l-k)(l+k+1)}{2}$. Napišme si n ve tvaru $n = 2^m a$, kde a je liché číslo. Pak $2^{m+1}a = (l-k)(l+k+1)$.

Jak $l-k$, tak $l+k+1$ jsou čísla větší než 1; jejich součet je lichý, takže jedno z nich je liché, a tedy a je větší než 1. Čísla tvaru 2^m proto nejdou vyjádřit jako součet několika po sobě jdoucích čísel.

Předpokládejme, že je naopak $a > 1$. Uvědom si, že pak je vždy jedno z čísel $a - 2^{m+1} - 1$ a $2^{m+1} - 1 - a$ nezáporné. V prvním případě můžeme volit $l = \frac{a+2^{m+1}-1}{2}$, $k = \frac{a-2^{m+1}-1}{2}$ a v druhém případě zase $l = \frac{a+2^{m+1}-1}{2}$ a $k = \frac{2^{m+1}-a-1}{2}$. Tato čísla podmínky zadání splňují, dostali jsme tedy vyjádření n jako součtu několika po sobě jdoucích čísel.

2. The left-hand side can be written as $(\frac{1}{3}f^3(x))'$. Then (by assumptions) $F(x) := \frac{1}{3}f^3(x) + \cos x$ has non-negative derivative on $\mathbb{R}_+$ and it is bounded on $\mathbb{R}_+$. Hence, $\lim_{x\to+\infty} F(x)$ exists. Taking $x_n := \pi/2 + n\pi$ and $y_n := 2n\pi$ one can see that $\lim_{x\to+\infty} f(x)$ does not exist.

3. **Solution:** If some 2×3 rectangle is covered by two corners, then we may remove all of the corners except those two. Thus, we may assume that no such rectangle exists.

We construct a directed graph whose vertices are the corners, as follows: for each corner, draw the 2×2 square containing that corner, and add an edge from this corner to the other corner covering the remainder of the 2×2 square. If one corner has no edge pointing toward it, we may remove that corner, so we may assume that no such corner exists. Hence, each edge of the graph is in some cycle. If there is more than one cycle, then we may remove all the corners except those in a cycle of minimal length, and the required property is preserved. Thus, it suffices to show that there cannot exist a single cycle consisting of all 111 vertices.

By the *center* of a corner we refer to the point at the center of the 2×2 square containing that corner. Recalling that we assumed that no two corners cover a 2×3 rectangle, one easily checks that if there is an edge pointing from one corner to another, then these corners' centers differ by 1 in both their x- and y- coordinates. Hence, in any cycle, the x-coordinates of the vertices in that cycle alternate, implying that the number of vertices in the cycle is even. Therefore, there cannot be a cycle containing all 111 vertices, as desired.

4. *Solution by John H. Smith, Needham, MA.* Let R be such a ring. For $x \in R$, let $xR = \{xz\colon z \in R\}$. We call xR *maximal* if it is not properly contained in yR for any y. By the hypothesis, every element of R is in some such set. Since R is finite, each yR is contained in a maximal such set.

When xR is maximal, we show that (i) $x \in xR$, (ii) xR contains an element e_x that acts as a multiplicative identity on xR, (iii) if yR is also maximal, then $xR = yR$, and (iv) $xR = R$. Together, (ii) and (iv) yield the desired multiplicative identity on R.

(i): We are given $x = yz$, which yields $xR \subseteq yR$. If $x \notin xR$, then the containment is proper, contradicting the maximality of xR.

(ii): Since $x \in xR$, we have $x = xe_x = e_x x$ (by commutativity) for some $e_x \in R$. Hence $xR \subseteq e_x R$, and these sets are equal by the maximality of xR. Thus also $e_x R$ is maximal, and by (i) we have $e_x \in e_x R = xR$. Since $e_x x = x$, it follows for $xy \in xR$ that $e_x xy = xy$, and hence e_x is a multipicative identity on xR.

(iii): Suppose that xR and yR are both maximal; let e_x and e_y be the corresponding identity elements, and let $f = e_x + e_y - e_x e_y$. We compute

$$fx = e_x x + e_y x - (e_x) e_y x = x + e_y x - e_y x = x.$$

Similarly, $fy = y$, so fR contains both xR and yR. By maximality, fR equals both xR and yR, and hence they equal each other.

(iv): Every element of R lies in some maximal set yR and hence in xR. Thus $R = xR$.