Řešení 1. soutěžní série

1. Suppose $f : \mathbb{Q} \to \{0, 1\}$ is a function with the property that for $x, y \in \mathbb{Q}$, if $f(x) = f(y)$ then $f(x) = f((x+y)/2) = f(y)$. If $f(0) = 0$ and $f(1) = 1$ show that $f(q) = 1$ for all rational numbers q greater than or equal to 1.

Solution:

Lemma. *Suppose that a and b are rational numbers. If $f(a) \neq f(b)$, then $f(n(b-a)+a) = f(b)$ for all positive integers n.*

Proof: We prove the claim by strong induction on n. For $n = 1$, the claim is clear. Now assume that the claim is true for $n \leq k$. Let $(x_1, y_1, x_2, y_2) = (b, k(b-a)+a, a, (k+1)(b-a)+a)$. By the induction hypothesis, $f(x_1) = f(y_1)$. We claim that $f(x_2) \neq f(y_2)$. Otherwise, setting $(x, y) = (x_1, y_1)$ and $(x, y) = (x_2, y_2)$ in the given condition, we would have $f(b) = f((x_1 + y_1)/2)$ and $f(a) = f((x_2 + y_2)/2)$. However, this is impossible because $x_1 + y_1 = x_2 + y_2$. Therefore, $f(y_2)$ must equal the value in $\{0, 1\} - \{f(a)\}$, namely $f(b)$. This completes the induction. ■

Applying the lemma with $a = 0$ and $b = 1$, we see that $f(n) = 1$ for all positive integers n. Thus, $f(1 + r/s) \neq 0$ for all natural numbers r and s, because otherwise applying the lemma with $a = 1$, $b = 1 + r/s$, and $n = s$ yields $f(1 + r) = 0$, a contradiction. Therefore, $f(q) = 1$ for all rational numbers $q \geq 1$.

2. *Solution by John Zacharias, Melbourne, FL.* By the Rearrangement Inequality, we have $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$. After adding 3 to each side and applying the AM–GM inequality, we get

$$\begin{aligned} a^3 + b^3 + c^3 + 3 &\geq (a^2b + 1) + (b^2c + 1) + (c^2a + 1) \\ &\geq 3((a^2b + 1)(b^2c + 1)(c^2a + 1))^{1/3}. \end{aligned}$$

3. Assume all three numbers are primes. Clearly none of them can be 2 otherwise it would divide an odd number. So we have: $a \mid (a+1)(b+1)(c+1) - 1$, $b \mid (a+1)(b+1)(c+1) - 1$, $c \mid (a+1)(b+1)(c+1) - 1$ and since a, b, c are distinct primes: $abc \mid (a+1)(b+1)(c+1) - 1$. Now note that:

$$1 < \frac{(a+1)(b+1)(c+1) - 1}{abc} < \frac{(a+1)(b+1)(c+1)}{abc} = \frac{a+1}{a} \cdot \frac{b+1}{b} \cdot \frac{c+1}{c} < \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} < 2$$

and thus $abc \nmid (a+1)(b+1)(c+1) - 1 \Rightarrow$ at least one of a, b, c is not a prime.

4. **Solution:** Suppose, for sake of contradiction that the sum of the numbers on the main diagonal is less than 1. Call a square *good* if its number is greater than the number in the square in the same column that lies on the main diagonal. Each row must contain a good square, because otherwise the numbers in that row would have sum less than 1.

For each square on the main diagonal, draw a horizontal arrow from that square to a good square in its row, and then draw a vertical arrow from that good square back to the main diagonal. Among these arrows, some must form a loop. We consider the following squares: the squares on the main diagonal which are not in the loop, and the good squares which are in the loop. Each row and column contains exactly one of these squares. However, the product of the numbers in these squares is greater than the product of the numbers on the main diagonal, a contradiction.