Řešení 5. soutěžní série

1. If A were regular, then by multiplying by A^{-1} one would obtain

$$B - A^{-1}BA = E.$$

However, conjugacy preserves the trace, hence the left-hand side is a traceless matrix. Since we work over a field of characteristic 0, the unit matrix has non-zero trace, a contradiction.

2.

$$\sum_k \binom{k}{m}\binom{n}{k} = \sum_k \binom{n}{m}\binom{n-m}{k-m} = \binom{n}{m}2^{n-m}.$$

3. For a rational number p/q expressed in lowest terms, define its *height* $H(p/q)$ to be $|p| + |q|$. Then for any $p/q \in S$ expressed in lowest terms, we have $H(f(p/q)) = |q^2 - p^2| + |pq|$; since by assumption p and q are nonzero integers with $|p| \neq |q|$, we have

$$\begin{aligned} H(f(p/q)) - H(p/q) &= |q^2 - p^2| + |pq| - |p| - |q| \\ &\geq 3 + |pq| - |p| - |q| \\ &= (|p| - 1)(|q| - 1) + 2 \geq 2. \end{aligned}$$

It follows that $f^{(n)}(S)$ consists solely of numbers of height strictly larger than $2n + 2$, and hence

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset.$$

Note: many choices for the height function are possible: one can take $H(p/q) = \max\{|p|, |q|\}$, or $H(p/q)$ equal to the total number of prime factors of p and q, and so on. The key properties of the height function are that on one hand, there are only finitely many rationals with height below any finite bound, and on the other hand, the height function is a sufficiently "algebraic" function of its argument that one can relate the heights of p/q and $f(p/q)$.

4.
It is well known that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8}.$$

Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\left(\frac{\pi}{4} + \sum_{k=1}^{n} \frac{(-1)^k}{2k-1}\right) \\ &= -\frac{\pi^2}{16} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=1}^{n} \frac{(-1)^k}{2k-1} \\ &= -\frac{\pi^2}{16} + \sum_{1 \leq k \leq n} \frac{(-1)^{n+k}}{(2k-1)(2n-1)} \\ &= -\frac{\pi^2}{16} + \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\right) \\ &= -\frac{\pi^2}{16} + \frac{1}{2}\left(\frac{\pi^2}{16} + \frac{\pi^2}{8}\right) = \frac{\pi^2}{32}. \end{aligned}$$