Řešení 3. soutěžní série

1. First observe that if $A \subseteq X$ satisfies $f(A) = A$, then for every $a \in A$, A contains the whole cycle of f containing a (f is a bijection on a finite set, and as such, it decomposes into cycles). Knowing this, we easily deduce that every A with $f(A) = A$ is the (disjoint) union of cycles of f, hence the number of such A's equals $2^{\text{number of cycles of } f}$.

Alternative solution(s). The system of subsets $A \subseteq X$ with the property from the statement is closed under unions, intersections, and complements, and as such, it is a (finite) Boolean algebra. However, cardinality of every finite Boolean algebra is a power of 2. The same can be derived by viewing the system in question as a vector space over the two-element field.

2. Parciální zlomky dávají

$$\frac{1}{k(k+\frac{1}{2})} = \frac{2}{k} - \frac{2}{k+\frac{1}{2}}.$$

Protože řada konverguje, stačí počítat limitu podposloupnosti částečných součtů s_{2N}:

$$\begin{aligned}\sum_{k=1}^{2N} \frac{1}{k(k+\frac{1}{2})} &= 2\left(1 + \frac{1}{2} + \dots + \frac{1}{2N} - \frac{2}{3} - \frac{2}{5} - \frac{2}{4N+1}\right) \\ &= 2\left(1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots - \frac{1}{2N+1}\right) - 4\left(\frac{1}{2N+3} + \dots + \frac{1}{4N+1}\right) \\ &= 2A_N - 4B_N.\end{aligned}$$

První závorka, tj. A_N, konverguje k $2 - \ln 2$ (známá řada pro logaritmus). Druhá závorka se odhadne pomocí integrálů:

$$\int_{N+1}^{2N+1} \frac{1}{2x+1} \le B_N \le \int_{N}^{2N} \frac{1}{2x+1},$$

tj.

$$\frac{1}{2}\ln\frac{4N+3}{2N+3} \le B_N \le \frac{1}{2}\ln\frac{4N+1}{2N+1},$$

takže $B_N \to \frac{1}{2}\ln 2$ a součet řady je $4 - 4\ln 2$.

3. Suppose that P does not have n distinct roots. Then it has a root of multiplicity at least 2, which we may assume is $x = 0$ without loss of generality. Let x^k be the greatest power of x dividing $P(x)$, so that $P(x) = x^k R(x)$ with $R(0) \neq 0$ a simple computation yields

$$P''(x) = (k^2 - k)x^{k-2}R(x) + 2kx^{k-1}R'(x) + x^k R''(x).$$

Since $R(0) \neq 0$ and $k \geq 2$, we conclude that the greatest power of x dividing $P''(x)$ is x^{k-2}. But $P(x) = Q(x)P''(x)$, and so x^2 divides $Q(x)$. We deduce (since Q is quadratic) that $Q(x)$ is a constant C times x^2. In fact, $C = 1/n(n-1)$ by inspection of the leading-degree terms of $P(x)$ and $P''(x)$. Now if $P(x) = \sum_{j=0}^{n} a_j x^j$, then the relation $P(x) = Cx^2P''(x)$ implies that $a_j = Cj(j-1)a_j$ for all j, hence $a_j = 0$ for $j \leq n-1$, and we conclude that $P(x) = a_n x^n$, which has all identical roots.

4.

Recall that $\operatorname{tr} XY^*$ is a scalar product on the space of complex $n \times n$ matrices (* denotes the Hermitean conjugation, i.e. transpose + complex conjugate). Therefore to show that A and B commute, we may check that the norm of their commutator is zero:

$$\begin{aligned}\operatorname{tr}(AB - BA)(AB - BA)^* &= \operatorname{tr}(AB - BA)(BA - AB) \\ &= \operatorname{tr} ABBA - \operatorname{tr} ABAB - \operatorname{tr} BABA + \operatorname{tr} BAAB = 0\end{aligned}$$

(we have used the fact that $\operatorname{tr} XY = \operatorname{tr} YX$, the property of A, B being Hermitean, and the relation from the statement).

Alternative solution. We may w.l.o.g. assume that A is diagonal with the (real) eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal (the corresponding change of basis is orthonormal, so B stays Hermitean). Then A^2 is diagonal with the diagonal $\lambda_1^2, \ldots, \lambda_n^2$, and since $B^2 = BB^*$, the diagonal of B^2 comprises of the Euclidean norms of the columns of B. Hence the i,i-th entry of A^2B^2 is

$$\lambda_i^2(|b_{i1}|^2 + \cdots + |b_{in}|^2).$$

A similar computation reveals that the i,i-th entry of $(AB)^2$ is

$$\lambda_i(\lambda_1|b_{i1}|^2 + \cdots + \lambda_n|b_{in}|^2).$$

Now we can examine the difference of the traces from the statement:

$$0 = \operatorname{tr}\bigl(A^2B^2 - (AB)^2\bigr) = \sum_{i=1}^{n} \lambda_i \Bigl(\sum_{j=1}^{n} (\lambda_i - \lambda_j)|b_{ij}|^2\Bigr) = \sum_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2 |b_{ij}|^2$$

(the equality $|b_{ij}| = |b_{ji}|$ was exploited). This being the sum of non-negative reals, we infer that $(\lambda_i - \lambda_j)b_{ij} = 0$ for all i, j. However, this is easily seen to be equivalent to $AB = BA$.