Řešení 2. soutěžní série

1. When we add a row of the matrix to another one, we get a row with elements from $\{0, \pm 2\}$. Expanding the determinant with respect to this new row, we get a sum of determinants of $(n-1) \times (n-1)$ matrices with elements ± 1, multiplied by the elements of the row. The rest follows by induction.

2. For a given π, no more than three different values of $\pi(x)$ are possible (four would require one part each of size at least 1,2,3,4, and that's already more than 9 elements). If no such $(x; y)$ exist, each pair $(\pi(x); \pi'(x))$ occurs for at most 1 element of x, and since there are only 3×3 possible pairs, each must occur exactly once. In particular, each value of $\pi(x)$ must occur 3 times. However, clearly any given value of $\pi(x)$ occurs $k\pi(x)$ times, where k is the number of distinct partitions of that size. Thus $\pi(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, contradiction.

3. The normality of H is clear. Therefore, we may instead check that the factor G/H is commutative, or equivalently, that the relation of being conjugate in G/H coincides with equality. To verify the latter, pick $x, y, a \in G$ such that

$$xH = aHyHa^{-1}H = aya^{-1}H.$$

This is the same as the existence of $h \in H$ such that $xh = aya^{-1}$. Seeing that xh and y are conjugate in G and taking into account the assumption on H, we may assume that $a \in H$. It follows that $aH = H$, or $xH = yH$, as desired.

4. Denote $s_n := \sum_{k=1}^{n} a_k^2$. Since $a_n s_n \to 1$, we can observe that $a_n \to 0$ and $s_n \to +\infty$. From $a_n s_n \to 1$ we have also $a_n s_{n-1} \to 1$. We apply Stolz theorem to $\frac{s_n^3}{n}$ (ratio of two sequences with limits $+\infty$). Since

$$\lim_{n\to\infty} \frac{s_n^3 - s_{n-1}^3}{n-(n-1)} = \lim_{n\to\infty} ((a_n^2 + s_{n-1})^3 - s_{n-1}^3) = \lim_{n\to\infty} (3a_n^2 s_{n-1}^2 + 3a_n^4 s_{n-1} + a_n^6) = 3 + 0 + 0,$$

we have $\lim_{n\to\infty} \frac{s_n^3}{n} = 3$, therefore $\lim_{n\to\infty} n a_n^3 = \frac{1}{3}$ and the proof is done.