Qualification Round — Category I

February 21, 2014, room T7

Problem I.1 Determine for which integers a the Diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

has infinitely many integer solutions (x, y, z) such that $gcd(a, xyz) = 1$. (10 points)

Problem I.2 Suppose a, b, c, x, y, z are positive numbers such that $a + b + c = x + y + z$ and $abc = xyz$. Show that if $\max\{x, y, z\} \geq \max\{a, b, c\}$, then $\min\{x, y, z\} \geq \min\{a, b, c\}$. (10 points)

Problem I.3 Let $A_1, \ldots, A_n$ be positive-definite 2×2 matrices of real numbers. Let G be the set of all unitary 2×2 complex matrices. Define $F : G^n \to \mathbb{R}$ by

$$F(U_1, ..., U_n) := \det\left(\sum_{k=1}^{n} U_k^* A_k U_k\right).$$

Show that

$$\min_{U \in G^n} F(U) = \sum_{k=1}^{n} \sigma_1(A_k) \cdot \sum_{k=1}^{n} \sigma_2(A_k),$$

where $\sigma_1(A_j)$ and $\sigma_2(A_j)$ denote the greatest and least eigenvalue of A_j, respectively. (10 points)

Problem I.4 Let $f_1 : (0, 1] \to \mathbb{R}$ and define $f_{n+1}(x) := x^{f_n(x)}$ for $x \in (0, 1]$ and $n = 1, 2, \ldots$. Denote $a_n := \lim_{x \to 0+} f_n(x)$ if it exists.

(a) Let m be such that a_m exists, $a_m \neq 0$. Prove that $|a_k - a_{k+1}| = 1$ for all $k \geq m + 2$.

(b) Does there exist f_1 such that $a_m = 0$ for all $m \in \mathbb{N}$?

(10 points)

If U is a matrix, then U^* denotes the transpose of the complex conjugate of U, i.e. $U^* = \bar{U}^T$. U is *unitary* if $U^*U = I$, where I is the identity matrix.