THEORY OF INTERPOLATION 1
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LUBOŠ PICK

1. Introduction

If not stated otherwise, $(\mathcal{R},\mu)$ and $(\mathcal{S},\nu)$ will throughout denote σ-finite measure spaces. By $\mathcal{M}(\mathcal{R},\mu)$ (or just $\mathcal{M}(\mathcal{R})$ for short in cases when it is clear which measure is considered) we denote the set of all μ-measurable real-valued functions on $\mathcal{R}$, and by $\mathcal{M}_+(\mathcal{R},\mu)$ the set of all nonnegative functions in $\mathcal{M}(\mathcal{R},\mu)$. For $p\in[1,\infty]$, we define p' by

$$p'=\begin{cases}\infty & \text{if } p=1,\\ \frac{p}{p-1} & \text{if } p\in(1,\infty),\\ 1 & \text{if } p=\infty.\end{cases}$$

If X and Y are (quasi)-normed spaces, we say that X is *embedded* into Y if there exists a constant C such that for every $x\in X$ one has $\|x\|_Y\le C\|x\|_X$. By $X+Y$ we denote the set of all elements z for which there exists a decomposition $z=x+y$ with $x\in X$ and $y\in Y$. We define the functional $\|\cdot\|_{X+Y}\colon (X+Y)\to[0,\infty]$ by $\|z\|_{X+Y}=\inf_{z=x+y}(\|x\|_X+\|y\|_Y)$.

Definition. The *Laplace transform* is defined by the formula

$$\mathcal{L}f(t)=\int_0^\infty f(s)e^{-st}\,ds \quad \text{for } t\in(0,\infty)$$

and every $f\in\mathcal{M}(0,\infty)$ for which the integral makes sense.

Remark. One has

$$\|\mathcal{L}f\|_{L^\infty(0,\infty)}\le\|f\|_{L^1(0,\infty)}.$$

Theorem 1 (Laplace transform on L^2)**.** *For every $f\in L^2(0,\infty)$ one has*

$$\|\mathcal{L}f\|_{L^2(0,\infty)}\le\sqrt{\pi}\|f\|_{L^2(0,\infty)}.$$

The constant is optimal.

Theorem 2 (embeddings of Lebesgue spaces)**.** *Let $0<p,q\le\infty$. Then the embedding*

$$L^q(\mathcal{R},\mu)\hookrightarrow L^p(\mathcal{R},\mu)$$

holds if and only if one of the following conditions hold:

- $p=q$,
- $p<q$ *and* $\mu(\mathcal{R})<\infty$,
- $p>q$ *and there is an* $\varepsilon>0$ *such that for every measurable* $E\subset\mathcal{R}$ *of positive measure one has* $\mu(E)\ge\varepsilon$.

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Theorem 3 (interpolation principle for Lebesgue spaces)**.** *Let* $0 < p < r < q \leq \infty$. *Assume that* $f \in L^p(\mathcal{R}, \mu) \cap L^q(\mathcal{R}, \mu)$. *Let* $\theta \in [0,1]$ *and let* r *be defined by*

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

Then $f \in L^r(\mathcal{R}, \mu)$ *and*

$$\|f\|_r \leq \|f\|_p^{1-\theta} \|f\|_q^{\theta}.$$

2. Classical interpolation theorems

2.1. **Interpolation of positive operators.**

Definition. Let T be an operator defined on simple functions on $(\mathcal{R}, \mu)$ with values in $\mathcal{M}(S, \nu)$. Let $p, q \in (0, \infty]$. We say that T is of *strong type* (p, q) if there exists a constant M such that

$$\|Tf\|_{L^q(S,\nu)} \leq M \|f\|_{L^p(\mathcal{R},\mu)} \quad \text{for every } \mu\text{-simple function } f.$$

The smallest such M is called the *norm* of T and it is denoted by $\|T\|_{L^p \to L^q}$.

Theorem 4 (interpolation of positive linear operators)**.** *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and* $\theta \in [0,1]$. *Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T *be a positive linear operator of the form*

$$Tf(y) = \int_R f(x) A(x,y) d\mu(x) \quad \text{for } y \in S,$$

where A *is a nonnegative measurable function on* $\mathcal{R} \times S$. *Assume that* T *is of strong type* (p_0, q_0) *and, at the same time, of strong type* (p_1, q_1) *with norms* M_0 *and* M_1, *respectively. Then* T *is of strong type* (p, q) *with norm* M_θ *satisfying*

$$M_\theta \leq M_0^{1-\theta} M_1^{\theta}.$$

2.2. **Riesz's-Thorin's interpolation theorem.**

Theorem 5 (Hadamard's three-line theorem)**.** *Let* F *be a bounded continuous function on* $\overline{\Omega}$ *and analytic in* Ω, *where*

$$\Omega = \{z \in \mathbb{C} \colon \operatorname{Re} z \in (0,1)\}.$$

Then the function M_θ, *defined by*

$$M_\theta = \sup\{|F(\theta + iy)| \colon y \in \mathbb{R}\} \quad \text{for } \theta \in [0,1],$$

satisfies

$$M_\theta \leq M_0^{1-\theta} M_1^{\theta} \quad \text{for } \theta \in [0,1].$$

Theorem 6 (the Riesz–Thorin interpolation theorem)**.** *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *and let* $\theta \in [0,1]$. *Let* T *be a linear operator which is of strong type* (p_0, q_0) *with norm* M_0 *and, at the same time, of strong type* (p_1, q_1) *with norm* M_1. *Suppose that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T *is of strong type* (p, q) *with norm* M_θ *satisfying*

$$M_\theta \leq 2 M_0^{1-\theta} M_1^{\theta}.$$

The constant 2 *can be dropped if the function spaces are complex.*

Definition. The *Fourier transform* is defined by the formula

$$\mathcal{F}f(x)=\int_{\mathbb{R}^n} f(y)e^{2\pi ixy}\,dy \quad \text{for } x\in\mathbb{R}^n$$

and every $f\in\mathcal{M}(\mathbb{R}^n)$ for which the integral makes sense.

Theorem 7 (the Hausdorff–Young theorem)**.** *Assume that $1\le p\le 2$. Then there exists a constant C such that*

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.1}$$

Theorem 8 (Young's convolution theorem)**.** *Let $p,q,r\in[1,\infty]$ and assume that*

$$\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1.$$

Then

$$\|f*g\|_{L^r(\mathbb{R}^n)}\le\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

2.3. **Interpolation of compact operators.**

Theorem 9 (interpolation of compact operators)**.** *Let $1\le p_0,p_1,q_0,q_1\le\infty$ and let T be a linear operator which is of strong type (p_0,q_0) and, at the same time, it is compact from $L^{p_1}(\mathcal{R},\mu)$ to $L^{q_1}(S,\nu)$. Let $\theta\in(0,1]$, $\nu(S)<\infty$, and suppose that*

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\quad\textit{and}\quad\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

Then T is compact from $L^p(\mathcal{R},\mu)$ to $L^q(S,\nu)$.

Corollary. *The Hardy operator T, defined by*

$$Tf(t)=\int_0^t f(s)\,ds\quad \textit{for } t\in(0,1)$$

for every $f\in\mathcal{M}(0,1)$ for which the integral makes sense, is compact from $L^q(0,1)$ to $L^\infty(0,1)$ for every $q\in(1,\infty]$.

2.4. **Interpolation of weak-type operators.**

Definition. Let $n\in\mathbb{N}$ and $\gamma\in(0,n)$. The *Riesz potential* I_γ is defined by the formula

$$I_\gamma f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\gamma}}\quad\text{for } x\in\mathbb{R}^n$$

and every function $f\in\mathcal{M}(\mathbb{R}^n)$ for which the integral makes sense.

Definition. Let $\delta>0$. The *dilation operator* τ_δ is defined by the formula

$$\tau_\delta f(x)=f(\delta x)\quad\text{for } x\in\mathbb{R}^n$$

and every function $f\in\mathcal{M}(\mathbb{R}^n)$.

Theorem 10 (weak type estimate for the Riesz potential)**.** *Let $n\in\mathbb{N}$ and $\gamma\in(0,n)$. Then there exists a constant C such that*

$$\sup_{\lambda\in(0,\infty)}\lambda|\{x\in\mathbb{R}^n\colon |I_\gamma f(x)|>\lambda\}|^{1-\frac{\gamma}{n}}\le C\|f\|_{L^1(\mathbb{R}^n)}$$

for every $f\in L^1(\mathbb{R}^n)$.

Definition. The *Hardy averaging operator* A is defined by the formula

$$Af(t)=\frac{1}{t}\int_0^t f(s)\,ds\quad\text{for } s\in(0,\infty)$$

and every function $f\in\mathcal{M}(0,\infty)$ for which the integral makes sense.

Remark. We have

$$\sup_{\lambda\in(0,\infty)} \lambda|\{x\in(0,\infty)\colon |Af(x)|>\lambda\}| \le \|f\|_{L^1(0,\infty)}$$

for every $f\in L^1(0,\infty)$.

Theorem 11 (interpolation of weak-type operators in the diagonal case)**.** *Let T be a quasilinear operator, that is, T is positively homogeneous and, moreover,*

$$|T(f+g)|\le K(|Tf|+|Tg|)$$

for some positive K and every f,g for which the right-hand side makes sense. Assume that there exists a constant C_∞ such that

$$\|Tf\|_{L^\infty(S,\nu)}\le C_\infty\|f\|_{L^\infty(\mathcal{R},\mu)}$$

for all $f\in L^\infty(\mathcal{R},\mu)$, and, at the same time, there exists a constant C_1 such that

$$\sup_{\lambda\in(0,\infty)} \lambda\nu(\{y\in S\colon |Tf(y)|>\lambda\})\le C_1\|f\|_{L^1(\mathcal{R},\mu)}$$

for all $f\in L^1(\mathcal{R},\mu)$. Then for every $p\in(1,\infty]$ there exists a constant C_p such that

$$\|Tf\|_{L^p(S,\nu)}\le C_p\|f\|_{L^p(\mathcal{R},\mu)}$$

for every $f\in L^p(\mathcal{R},\mu)$ and

$$C_p\le 2KC_1^{\frac1p}C_\infty^{1-\frac1p}\left(\frac{p}{p-1}\right)^{\frac1p}.$$

Definition. Let $n\in\mathbb{N}$ and $\gamma\in[0,n)$. The *fractional maximal operator M_γ* is defined by the formula

$$M_\gamma f(x)=\sup_{Q\ni x}\frac{1}{|Q|^{1-\frac{\gamma}{n}}}\int_Q|f(y)|\,dy \quad \text{for } x\in\mathbb{R}^n$$

and every function $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where the supremum is extended over all cubes with sides parallel to coordinate axes. In particular, M_0 is the *Hardy–Littlewood maximal operator*.

Theorem 12 (Vitali-type covering theorem)**.** *Let $\mathcal{B}$ be a finite collection of balls in $\mathbb{R}^n$. Then there is a disjoint subcollection $\mathcal{B}'$ of $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{B}}B\subset\bigcup_{B\in\mathcal{B}'}3B. \tag{2.2}$$

Theorem 13 (weak-type estimate for the Hardy–Littlewood maximal operator)**.** *One has*

$$\sup_{\lambda\in(0,\infty)}\lambda|\{x\in\mathbb{R}^n\colon Mf(x)>\lambda\}|\le\|f\|_{L^1(0,\infty)} \tag{2.3}$$

for every $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

2.5. **Distribution function and nonincreasing rearrangement.**

Definition. Let $f\colon(\mathcal{R},\mu)\to\mathbb{R}$ be a measurable function. Then the function $f_*\colon[0,\infty)\to[0,\infty]$, defined by

$$f_*(\lambda)=\mu\left(\{x\in\mathcal{R}:|f(x)|>\lambda\}\right)\quad\text{for }\lambda\in[0,\infty), \tag{2.4}$$

is called the *distribution function* of f.

Definition. Let $f\in\mathcal{M}(\mathcal{R},\mu)$. Then the function $f^*\colon[0,\infty)\to[0,\infty]$ defined by

$$f^*(t)=\inf\{\lambda>0:f_*(\lambda)\le t\}\quad\text{for }t\in[0,\infty),$$

is called the *nonincreasing rearrangement* of f.

Observation. Assume that f is a measurable function. Then the following statements hold.

(a) for every $s, t > 0$, one has

$$s < f^*(t) \quad \Leftrightarrow \quad t < f_*(s),$$

(b) $(f^*)_* = f_*$,

(c) $(f_*)_* = f^*$,

(d) f_* and f^* are nonnegative, nonincreasing and right continuous (and thus lower semicontinuous) on $(0, \infty)$,

(e) one has

$$\int_{\mathcal{R}} |f| d\mu = \int_0^\infty f_*(s)\, ds.$$

Proposition. *For every measurable function f, one has*

$$\int_{\mathcal{R}} |f| d\mu = \int_0^\infty f_*(s)\, ds = \int_0^\infty f^*(t)\, dt.$$

Remark. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$ and $p \in (0, \infty)$. Then

$$\int_R |f(x)|^p\, d\mu = \int_0^\infty f^*(t)^p\, dt.$$

Theorem 14 (basic estimates for f^* and f_*)**.** *Let f, g be measurable functions and let $\lambda, s, t > 0$. Then*

(a) $f^*(f_*(\lambda)) \le \lambda$,

(b) $f_*(f^*(t)) \le t$,

(c) $(f+g)^*(s+t) \le f^*(s) + g^*(t)$.

Theorem 15 (the Hardy–Littlewood inequality)**.** *For every $f, g \in \mathcal{M}(\mathcal{R}, \mu)$, one has*

$$\int_{\mathcal{R}} f(x) g(x)\, d\mu \le \int_0^\infty f^*(t) g^*(t)\, dt.$$

Definition. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$. The we define the *maximal nonincreasing rearrangement*, f^{**}, of f, by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds, \quad \text{for } t \in (0, \infty).$$

Remark. For every $f \in \mathcal{M}_0(\mathcal{R}, \mu)$, the function f^{**} is nonincreasing on $(0, \infty)$ and one has $f^*(t) \le f^{**}(t)$ for every $t \in (0, \infty)$.

Definition. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$ and $t \in (0, \mu(\mathbb{R}))$. We say that a measurable function g is a *test function* for $f^{**}(t)$ if

$$0 \le g \le \frac{1}{t} \quad \text{and} \quad \int_{\mathcal{R}} g\, d\mu = 1.$$

The collection of all test functions will be denoted by $\mathcal{G}_t$.

Theorem 16 (characterization of f^{**})**.** *For every $f, g \in \mathcal{M}(\mathcal{R}, \mu)$ and $t > 0$, one has*

$$f^{**}(t) = \sup_{g \in \mathcal{G}_t} \int_{\mathcal{R}} |f| g\, d\mu.$$

Corollary. *The operation $f \mapsto f^{**}$ is subadditive.*

2.6. **Lorentz spaces.**

Definition. Let $f \in \mathcal{M}(\mathcal{R}, \mu)$ and $g \in \mathcal{M}(S, \nu)$. We say that f and g are *equimeasurable* if they have the same distribution function, that is, if $f_*(\lambda) = g_*(\lambda)$ for all $\lambda \in [0, \infty)$. We write $f \sim g$.

Definition. Assume that $p, q \in (0, \infty]$. We define the functional $\|\cdot\|_{p,q} \colon \mathcal{M}(\mathcal{R}, \mu) \to [0, \infty]$ by

$$\|f\|_{p,q} = \|t^{\frac{1}{p} - \frac{1}{q}} f^*(t)\|_{L^q(0,\infty)}.$$

In other words, we have

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup\limits_{0<t<\infty} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

The collection of all functions $f \in \mathcal{M}(\mathcal{R}, \mu)$ such that $\|f\|_{p,q} < \infty$ is called the *Lorentz space* and is denoted by $L^{p,q}(\mathcal{R}, \mu)$.

Theorem 17 (elementary properties of Lorentz spaces)**.** *For every $p, q \in (0, \infty]$, the Lorentz space $L^{p,q}$ is a linear set and the functional $\|\cdot\|_{p,q}$ is a quasinorm and an α-norm on $L^{p,q}$. Moreover, one has*

$$\|f\|_{p,q} = p \|f_*(\lambda)^{\frac{q}{p}} \lambda^{1-\frac{1}{q}}\|_{L^q(0,\infty)} \quad \textit{for every } f \in \mathfrak{M}(\mathcal{R}, \mu).$$

Theorem 18 (Hanička's theorem)**.** *Let $f \colon (0, \mu(\mathcal{R})) \to [0, \infty]$ be right continuous and nonincreasing. Then, for every fixed $t \in (0, \mu(\mathcal{R}))$, the operator*

$$f \mapsto \int_0^t f^*(s) h(s)\, ds$$

is subadditive on $\mathfrak{M}_+$ and also on $\mathfrak{M}_0$.

Theorem 19 (on Lorentz norms)**.** *If $1 \le q \le p \le \infty$, then $\|\cdot\|_{p,q}$ is a norm.*

Theorem 20 (embeddings of Lorentz spaces)**.** *Let $p, q, r \in [0, \infty]$ be such that $q \le r$. Then $L^{p,q} \hookrightarrow L^{p,r}$.*

Definition. Assume that $p, q \in (0, \infty]$. We define the functional $\|\cdot\|_{(p,q)} \colon \mathcal{M}(\mathcal{R}, \mu) \to [0, \infty]$ by

$$\|f\|_{(p,q)} = \|t^{\frac{1}{p} - \frac{1}{q}} f^{**}(t)\|_{L^q(0,\infty)}.$$

Theorem 21 (Minkowski's integral inequality)**.** *Let $(\mathcal{R}, \mu)$ and (S, ν) be σ-finite measure spaces. Let $p \in [1, \infty)$ and let $F \colon (\mathcal{R} \times S) \to \mathbb{R}$ be measurable with respect to $\mu \times \nu$. Assume that*

$$\int_S \left(\int_R |F(x,y)|^p \, d\mu(x) \right)^{\frac{1}{p}} d\nu(y).$$

Then $\int_S F(x,y) d\nu(y)$ converges for μ-a.e. $x \in \mathcal{R}$ and

$$\left(\int_{\mathcal{R}} \left| \int_S F(x,y) d\nu(y) \right|^p d\mu(x) \right)^{\frac{1}{p}} \le \int_S \left(\int_R |F(x,y)|^p \, d\mu(x) \right)^{\frac{1}{p}} d\nu(y).$$

Theorem 22 (weighted Hardy's inequality)**.** *Let $1 < p < \infty$ and $f \in \mathcal{M}_+(0, \infty)$. If $\alpha < p - 1$, then*

$$\int_0^\infty \left(\frac{1}{t} \int_0^t f(s)\, ds \right)^p t^\alpha \, dt \le \left(\frac{p}{p - \alpha - 1} \right)^p \int_0^\infty f(t)^p t^\alpha \, dt.$$

Theorem 23 (alternative norm in a Lorentz space)**.** *Assume that $p \in (1, \infty]$ and $q \in [1, \infty]$. Then $\|\cdot\|_{(p,q)}$ is a norm. Moreover, the functionals $\|\cdot\|_{(p,q)}$ and $\|\cdot\|_{p,q}$ are equivalent in the sense that there exists a constant C such that*

$$\|f\|_{p,q} \le \|f\|_{(p,q)} \le C \|f\|_{p,q} \quad \textit{for every } f \in \mathcal{M}(\mathcal{R}, \mu).$$

2.7. Marcinkiewicz interpolation theorem.

Definition. Let $p \in [1,\infty)$, $q \in [1,\infty]$ and let T be an operator defined on $L^{p,1}(\mathcal{R},\mu)$ and taking values in $\mathcal{M}(S,\nu)$. Then T is said to be of *weak type* (p,q) if it is a bounded operator from $L^{p,1}(\mathcal{R},\mu)$ into $L^{q,\infty}(S,\nu)$, that is, if there exists a constant M such that

$$\|Tf\|_{q,\infty} \le M\|f\|_{p,q} \quad \text{for every } f \in L^{p,1}(\mathcal{R},\mu).$$

The lease such constant M is called the *weak-type* (p,q) *norm* of T. We say that T is of weak type (∞,q) if it is a bounded operator from $L^{\infty}(\mathcal{R},\mu)$ into $L^{q,\infty}(S,\nu)$.

Theorem 24 (Marcinkiewicz's interpolation theorem)**.** *Let $1 \le p_0 < p_1 < \infty$, $1 \le q_0, q_1 \le \infty$, $q_0 \ne q_1$, $0 < \theta < 1$ and $1 \le r \le \infty$. Let p, q be defined by the formulas*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Let T be a quasilinear operator defined on $(L^{p_0,1} + L^{p_1,1})(\mathcal{R},\mu)$ and taking values in $\mathcal{M}(S,\nu)$. Let T be of weak types (p_0,q_0) and (p_1,q_1) with respective weak-type norms M_0 and M_1. Then $T\colon L^{p,r} \to L^{q,r}$. More precisely, there exists a constant C such that

$$\|Tf\|_{q,r} \le \frac{C\max\{M_0,M_1\}}{\theta(1-\theta)}\|f\|_{p,r}.$$

Remarks. (a) Theorem 24 holds also in the case $p_1 = \infty$ provided that the hypothesis "of weak type (p_1,q_1)" is replaced by "of strong type (p_1,q_1)".

(b) If $p_i \le q_i$, $i = 0,1$, then it follows under the hypotheses of Theorem 24 that T is of strong type (p,q).

(c) The assumption $q_0 \ne q_1$ cannot be omitted. For instance, let α be a bounded linear functional on $L^1(0,1)$ and let the operator T be defined on $L^1(0,1)$ by

$$Tf(t) = \alpha(f)\frac{1}{\sqrt{t}} \quad \text{for } t \in (0,1).$$

Then T is of weak type $(1,2)$ and of weak type $(\infty,2)$, but it is not of strong type $(2,2)$.

Example. Assume that $1 \le p \le 2$. Then there exists a constant C depending on n and p such that

$$\|\mathcal{F}f\|_{L^{p',p}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for every } f \in L^p(R^n),$$

where $\mathcal{F}$ denotes the Fourier transform. Note that, thanks to Theorem 19, this is a better estimate than (2.1).

Example. Let $n \in \mathbb{N}$, $\gamma \in (0,n)$ and $p \in (1, \frac{n}{n-\gamma})$. Then there exists a constant C depending on n, p and γ such that

$$\|I_\gamma f\|_{L^{\frac{np}{n-p},p}(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for every } f \in L^p(R^n),$$

where I_γ denotes the Riesz potential.

Example. Let $p \in (1,\infty]$. Then there exists a constant C depending on p such that

$$\|Af\|_{L^p(0,\infty)} \le C\|f\|_{L^p(0,\infty)} \quad \text{for every } f \in L^p(0,\infty),$$

where A denotes the Hardy averaging operator.

Example. Let $p \in (1,\infty]$. Then there exists a constant C depending on p such that

$$\|\mathcal{L}f\|_{L^{p',p}(0,\infty)} \le C\|f\|_{L^p(0,\infty)} \quad \text{for every } f \in L^p(0,\infty),$$

where $\mathcal{L}$ denotes the Laplace transform.

Definition. The *Hilbert transform* H is defined by the formula

$$Hf(x) = p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \quad \text{for } x \in \mathbb{R}$$

and every function $f \in \mathcal{M}(\mathbb{R})$ for which the integral makes sense.

Example. Let $p \in (1, \infty)$. Then there exists a constant C depending on p such that

$$\|Hf\|_{L^p(0,\infty)} \leq C\|f\|_{L^p(0,\infty)} \quad \text{for every } f \in L^p(0, \infty),$$

where H denotes the Hilbert transform.

Email address: pick@karlin.mff.cuni.cz
ORCiD: 0000-0002-3584-1454

[3]Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic